Algebraic Topology I, Spring 2022.

Wed/Fri 1:30-3:05 理科楼 A112

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Course description. This is a first course on algebraic topology at the graduate level. We will cover homology and homotopy theory, compute basic examples, and discuss applications. Topics include singular and cellular homology groups, cohomology rings, Lefschetz fixed-point theorem, Poincare duality, homotopy groups.

Prerequisites. Abstract Algebra and Topology. We will assume the students are familiar with groups, rings, modules, point-set topology, covering spaces, and fundamental groups.

Grading. 30% homework + 30% midterm exam + 40% final exam.

Homework. Homework will be assigned weekly. The two lowest homework scores will be dropped at the end. Students are encouraged to work in group on homework assignment. They can also use books, papers, and internet. However, each student must write or type up her/his own solution and include an acknowledgement.

Textbook. Allen Hatcher, *Algebraic Topology*. The book is available for free on the author's webpage: https://pi.math.cornell.edu/~hatcher/AT/AT+.pdf

Make-up policy. Since students can miss two homework with no penalty, we will not make up for single missed homework. If a student is sick or away for more than two weeks, please contact me to arrange for make-up works.

Resources for learning. Your best resources for learning are the people around: your classmates, TA, and instructor. Join the class WeChat group for questions, discussions, and announcements. Come to office hours often!

Academic Integrity. Any form of cheating or plagiarism will not be tolerated and will be reported to the department and the university.

Algebraic Topology 1.

Before class:

① language for class?

② introduce self & TA.

③ go through syllabus

What is algebraic topology?

Topology: topological spaces
and continuous maps.

Basic questions:

Q1: How do we know if X and Y are homeomorphic or homotopy equivalent?

Q2: How do we know if certain continuous map $X \xrightarrow{f} Y$ exists?

Idea: Study questions using algebra
(numbers, groups, rings, vector spaces and homomorphisms).

Example: Fundamental group

$$\left\{ \begin{array}{c}\text{topological spaces} \\ \text{with a base pt}\end{array} \right\} \longrightarrow \{\text{groups}\}$$

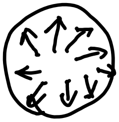
$$(X, x) \longmapsto \pi_1(X, x)$$

$$\pi_1(X,x) := \{ \gamma : [0,1] \to X \ / \ \gamma(0) = \gamma(1) = x \} \Big/ \begin{array}{l}\text{homotopy} \\ \text{rel base} \\ \text{pt.}\end{array}$$

Using π_1, we can show that

Thm. (Brouwer fixed pt thm).

$\nexists$ retraction $r : D^2 \to S^1$



(i.e. $\nexists$ continous $r : D^2 \longrightarrow S^1 = \partial D^2$

s.t. $r(x) = x \ \ \forall x \in S^1$).

Pf: If so,

$$\begin{tikzcd} S^1 \arrow[r, hook, "i"] \arrow[rr, bend left, "id"] & D^2 \arrow[r, "r"] & S^1 \end{tikzcd}$$

apply π_1 :

$$\begin{tikzcd} \pi_1(S^1) \arrow[r, "i_*"] \arrow[rr, bend left, "id"] \arrow[d, equal] & \pi_1 D^2 \arrow[r, "r_*"] \arrow[d, equal] & \pi_1(S^1) \arrow[d, equal] \\ \mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

□.

Limit of π_1:

Cannot distinguish S^2 and S^n $(n \geq 2)$.

(Brouwer's thm for $D^{n>2}$?)

For that, we need homology groups.

(Poincaré, Analysis Situs 1895).

There are many ways to define homology:

simplicial homology

singular homology

cellular homology

⋮

Simplicial homology.

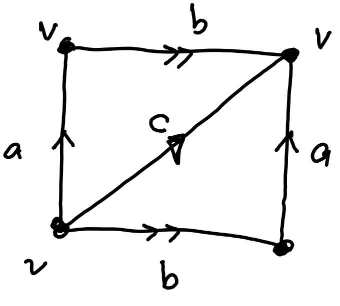
Reference: Hatcher 2.1.

Idea. a space X "with a Δ-complex structure"

$\xrightarrow{\text{simplicial homology}}$ $H_n(X)$ abelian groups.

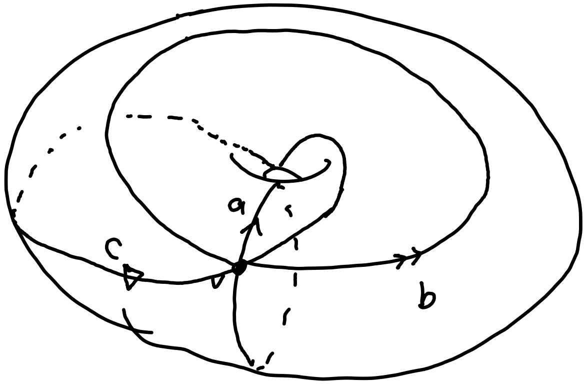
(topology $\rightsquigarrow$ combinatorics $\rightsquigarrow$ algebra).

Example 1:

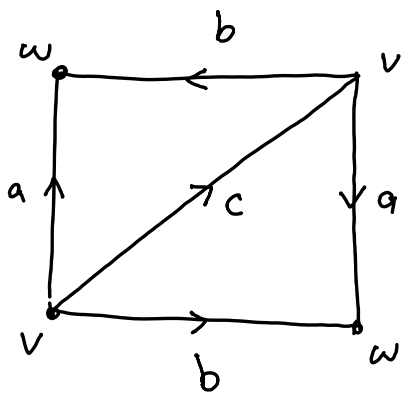


$\parallel$

T^2.



Example 2:

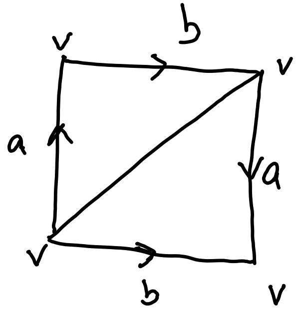


ss

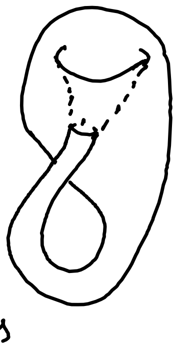
$$\mathbb{R}P^2 := \{\mathbb{R}\text{-lines in } \mathbb{R}^3\}$$

$$= \mathbb{R}^3 \setminus 0 \big/ \mathbb{R}^\times = S^2 \big/ \vec{v} \sim -\vec{v}.$$

Ex 3:



$\approx$ "Klein bottle".



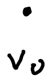
These spaces made by gluing triangles are called Δ-complexes.

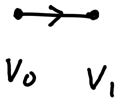
. **Def**: An **n-simplex** is the smallest convex set containing $n+1$ points $v_0, v_1, \dots, v_n$ in general position.

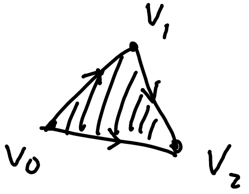
Notation: $[v_0, v_1, \dots, v_n]$

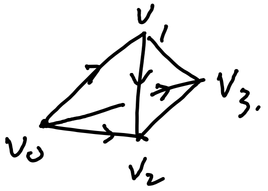
note: ordering on vertices

Ex:









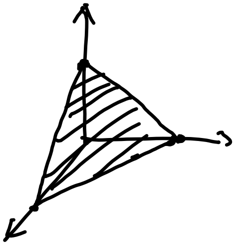
...

The **standard n-simplex** is

$$\Delta^n := \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} \sum t_i = 1 \\ t_i \geqslant 0 \quad \forall i \end{array} \right\}$$

Ex:

$n = 2$.



Remark: Every n-simplex $[v_0, \cdots, v_n]$ is homeomorphic to the standard n-simplex via:

$$\Delta^n \longrightarrow [v_0, \cdots, v_n]$$

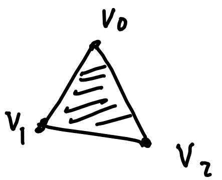
$$(t_0, \cdots, t_n) \longmapsto \sum_i t_i v_i$$

Def: The $(n-1)$-simplex $[v_0, \cdots \hat{v_j} \cdots v_n]$ is a face of $[v_0, \cdots, v_n]$.

The union of all faces of Δ^n is the boundary, written $\partial \Delta^n$.

The open simplex $\mathring{\Delta}^n$ is $\Delta^n \setminus \partial \Delta^n$.

Ex: $n = 2$.



Def: A <u>Δ-complex structure</u> on a space X is a collection of maps

$$\sigma_\alpha : \Delta^n \longrightarrow X$$

s.t.

(i) $\sigma_\alpha \big|_{\overset{\circ}{\Delta}{}^n}$ is injective

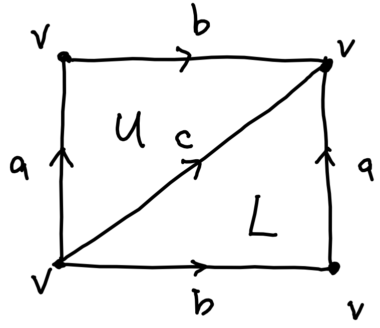
Each point of X is in the image of exactly one such $\sigma_\alpha \big|_{\overset{\circ}{\Delta}{}^n}$

(ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \longrightarrow X$.

(iii) $A \subseteq X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n, $\forall \alpha$.

Rmk: We can think "$X \approx \bigcup\limits_\alpha \sigma_\alpha$" ← simplices.

Ex:



$\sigma_v : \Delta^0 \longrightarrow X$

$v_0 \longmapsto v$

$\sigma_U : \Delta^2 \longrightarrow X$

$\sigma_L : \Delta^2 \longrightarrow X$

$\sigma_a : \Delta^1 \longrightarrow X$

σ_b

σ_c

T^2 has a Δ-complex structure.

(later).

Rmk: $\left\{\begin{array}{c}\text{simplicial}\\ \text{complexes}\end{array}\right\} \subseteq \left\{\Delta\text{-complexes}\right\} \subseteq \left\{\begin{array}{c}\text{CW}\\ \text{complexes}\end{array}\right\}$.

Simplicial homology.

Suppose X is a space with a Δ-complex structure.

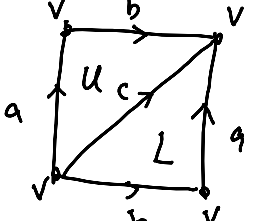
$C_n(X) :=$ the free abelian group with basis the n-simplices of X

(Hatcher uses $\Delta_n(X)$)

$$= \left\{ \sum_\alpha n_\alpha \sigma_\alpha \;\middle|\; \begin{array}{l} n_\alpha \in \mathbb{Z}, \\ \sigma_\alpha \text{ is an } n\text{-simplex of } X \end{array} \right\}$$

"n-chain"

Ex:

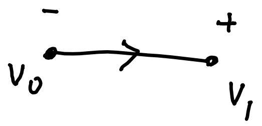


$C_0(T^2) = \mathbb{Z}\{v\} \cong \mathbb{Z}$

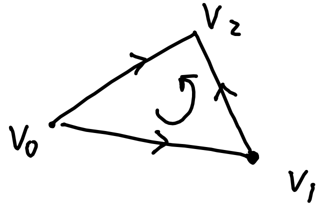
$C_1(T^2) = \mathbb{Z}\{a,b,c\} \cong \mathbb{Z}^3$

$C_2(T^2) = \mathbb{Z}\{U,L\} \cong \mathbb{Z}^2.$

<u>observe</u>: orientation on boundary of Δ^n.



$$\partial [v_0, v_1] = [v_1] - [v_0]$$



$$\partial [v_0, v_1, v_2]$$

$$= [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$$

⋮

$\boxed{\text{STOP}}$

The **boundary homomorphism** is

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

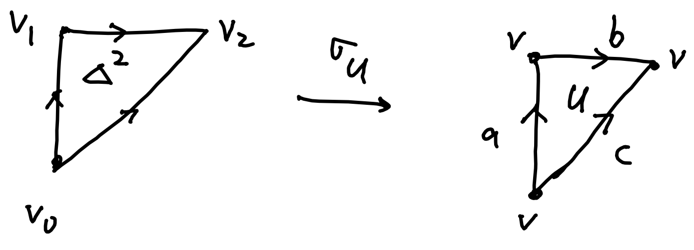
s.t. on basis:

$$\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \Big|_{[v_0, \cdots \hat{v_i} \cdots v_n]} .$$

Ex: Back to T^2.

$$\partial_1(\sigma_a) = \sigma_v - \sigma_v = 0$$

$$\begin{aligned}\partial_2(\sigma_u) &= \sigma|_{[v_1 v_2]} - \sigma|_{[v_0 v_2]} + \sigma|_{[v_0 v_1]} \\ &= \sigma_b - \sigma_c + \sigma_a\end{aligned}$$



Lemma: The composition

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$$

is zero.

Pf:

$$\partial_{n-1}\,\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \, \sigma\Big|_{[v_0 \cdots \hat{v}_j \cdots \hat{v}_i \cdots v_n]}$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \, \sigma\Big|_{[v_0 \cdots \hat{v}_i \cdots \hat{v}_j \cdots v_n]}$$

$$= 0.$$

□.

Lemma ⇒ $(C_n, \partial_n)_n$ forms a <u>chain complex</u> called "simplicial chain complex".

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r, "\partial_0"] & 0
\end{tikzcd}$$

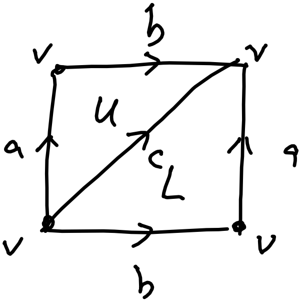
with $\partial_{n+1} \circ \partial_n = 0$.

Hence $$\underset{\{\text{boundaries}\}}{\underset{\parallel}{\mathrm{Im}\,\partial_{n+1}}} \subseteq \underset{\{\text{cycles}\}}{\underset{\parallel}{\mathrm{Ker}\,\partial_n}}$$

<u>**Def**</u>: The n-th <u>simplicial homology group</u> of X

is $$H_n^{\Delta}(X) := \frac{\mathrm{Ker}\,\partial_n}{\mathrm{Im}\,\partial_n} = \frac{\{n\text{-cycles}\}}{\{n\text{-boundaries}\}}.$$

Ex: $X = T^2$.



$C_0 = \mathbb{Z}\{v\}$

$C_1 = \mathbb{Z}\{a, b, c\}$

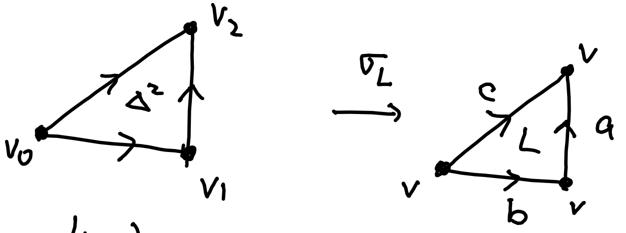
$C_2 = \mathbb{Z}\{U, L\}$

$$\begin{tikzcd} C_2 \arrow[r, "\partial_2"] \arrow[d, equal] & C_1 \arrow[r, "\partial_1"] \arrow[d, equal] & C_0 \arrow[r] \arrow[d, equal] & 0 \\ \mathbb{Z}^2 & \mathbb{Z}^3 & \mathbb{Z} & \end{tikzcd}$$

we computed $\partial_1(a) = \partial_1(b) = \partial_1(c) = v - v = 0$.

$\partial_2(U) = a + b - c$

$\partial_2(L) = a - c + b$



$H_2^{\Delta}(T^2) = \dfrac{\ker \partial_2}{0} = \mathbb{Z}\{U - L\} \cong \mathbb{Z}$

$H_1^{\Delta}(T^2) = \dfrac{\ker \partial_1}{\operatorname{Im} \partial_2} = \dfrac{\mathbb{Z}\{a, b, c\}}{\mathbb{Z}\{a+b-c\}} \cong \mathbb{Z}\{a, b\} \cong \mathbb{Z}^2$
$(c = a + b)$.

$H_0^{\Delta}(T^2) = \mathbb{Z}\{v\}. \cong \mathbb{Z}$

<u>explain</u>: $H_n(X) = \dfrac{\{n\text{-cycles}\}}{\{n\text{-boundaries}\}}$

<u>Q</u>: Are $H_n^{\Delta}(X)$ independent of the choice of Δ-complex structure on X?

To answer, we need "singular homology".

Singular homology.

Suppose X is a topological space

A singular n-simplex of X is a continuous map $\sigma : \Delta^n \to X$.

Define

$C_n(X) :=$ free abelian group with basis the set of singular n-simplices of X

$= \mathbb{Z}\{\sigma \mid \sigma : \Delta^n \to X \text{ continuous}\}$

infinite rank! (unlike simplicial chain complex)

Define boundary map:

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

by

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \Big|_{[v_0 \dots \hat{v_i} \dots v_n]} .$$

__note__: implicitly, we have

$$\Delta^{n-1} \xrightarrow{\cong} [v_0, \dots, \hat{v_i}, \dots v_n] \xrightarrow{\sigma|_{[v_0 \dots \hat{v_i} \dots v_n]}} X$$

__Lemma__: $\partial_{n-1} \partial_n = 0$

pf same as before.

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots \to C_0 \to 0$$

is a chain complex, called the "singular chain complex"

Def: The singular homology group of X

is $$H_n(X) := \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Rmk: (Compare simplicial homology v.s. singular homology)

Suppose X is a finite Δ-complex.

1. $H_n^{\Delta}(X)$ is finite rank
   Not clear for $H_n(X)$.

2. $H_n^{\Delta}(X) = 0$ if $n > \dim(X)$.
   Not clear for $H_n(X)$.

3. If X and Y are homeomorphic
   then $H_n(X) \cong H_n(Y)$ $\forall n$.
   Not clear for H_n^{Δ}.

Properties of singular homology.

Prop : If $X = \coprod_{\alpha} X_\alpha$ ↪ path components

then $H_n(X) = \bigoplus_{\alpha} H_n(X_\alpha)$

Pf: $\sigma : \Delta^n \to X$ has path connected image

$$C_n(X) = \bigoplus_{\alpha} C_n(X_\alpha)$$

∂_n preserves direct sum decomposition

i.e. $\partial_n : C_n(X_\alpha) \longrightarrow C_{n-1}(X_\alpha)$.

$\operatorname{Ker}\partial_n$. $\operatorname{Im}\partial_{n+1}$ splits as direct sums

so does $H_n(X) = \dfrac{\operatorname{Ker}\partial_n}{\operatorname{Im}\partial_{n+1}}$ $\square$.

Hence, we can focus on X path connected.

prop. If X is nonempty and path connected then $H_0(X) \cong \mathbb{Z}$.

pf: $H_0(X) = \dfrac{C_0(X)}{im\,\partial_1}$

Define $\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$

s.t. $\varepsilon(\sigma) = 1 \quad \forall\, \sigma : \underset{\{v_0\}}{\overset{}{\underset{\shortparallel}{\Delta^0}}} \to X$.

X nonempty $\Rightarrow$ ε is onto.

claim: $\ker \varepsilon = Im\,\partial_1$.

pf: ($Im\,\partial_1 \subseteq \ker \varepsilon$)

$\sigma : \Delta^1 = [v_0, v_1] \longrightarrow X$.

$\partial_1(\sigma) = \sigma|_{[v_1]} - \sigma|_{[v_0]}$

$\varepsilon\,\partial_1(\sigma) = 1 - 1 = 0$.

($\ker \varepsilon \subseteq Im\,\partial_1$)

Suppose $\varepsilon(\sum_i n_i \sigma_i) = 0$ so $\sum_i n_i = 0$.

σ_i's are pts in X.

$X \neq \emptyset \Rightarrow \exists\, x_0 \in X.$

X path connected $\Rightarrow$ $\exists$ a path $\tau_i : \overset{\Delta^1}{\overset{\shortparallel}{[0,1]}} \to X$

from x_0 to σ_i

$$\partial_1(\tau_i) = \tau_i\big|_{[1]} - \tau_i\big|_{[0]} = \sigma_i - x_0 .$$

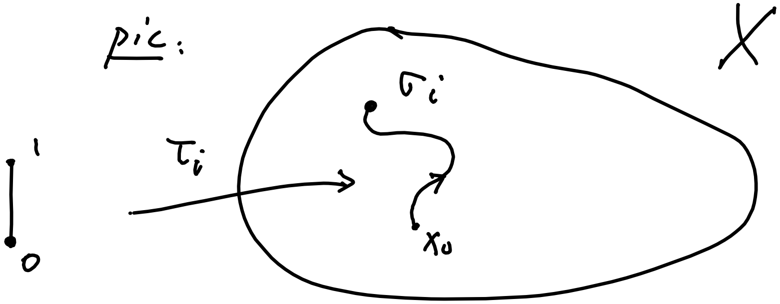
So
$$\partial_1\Big(\sum_i n_i \tau_i\Big) = \sum_i n_i(\sigma_i - x_0)$$

$$= \sum_i n_i \sigma_i - \underset{\overset{\shortparallel}{0}}{\Big(\sum_i n_i\Big)} x_0$$

$$= \sum_i n_i \sigma_i \in \operatorname{Im} \partial_1 .$$

$\square$.

pic:



prop. If X is a point.

then $H_n(X) = 0 \quad \forall n > 0$

$H_0(X) \cong \mathbb{Z}$.

Pf: exercise

Reduced homology groups.

Add one term to the singular chain complex

$$\cdots \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0.$$

note: ($\operatorname{Im}\partial_1 \subset \ker \varepsilon$ for any X nonempty).

Define $\tilde{H}_0(X) := \dfrac{\ker \varepsilon}{\operatorname{Im}\partial_1}$

$\tilde{H}_n(X) := H_n(X) \quad \forall n > 0.$

we have a SES $\quad 0 \to \tilde{H}_n(X) \to H_n(X) \to \mathbb{Z} \to 0.$

Hence, $\quad H_n(X) \cong \tilde{H}_n(X) \oplus \mathbb{Z}$.

Naturality.

Thm: A continuous map $f: X \to Y$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$ $\forall n$.

(Algebra)

$(C_\bullet, \partial)$

A chain complex is a sequence of abelian groups C_n, $n \in \mathbb{Z}$ with homomorphisms $\partial_n: C_n \to C_{n-1}$ s.t. $\partial_{n-1} \circ \partial_n = 0$.

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

Suppose (C, ∂) and (C', ∂') are chain complexes.

A chain map $(C, \partial) \xrightarrow{f} (C', \partial')$ consists of maps

$C_n \xrightarrow{f_n} C_n'$ s.t. $f\partial = \partial' f$

i.e. the diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n' \arrow[r, "\partial_{n-1}"] & C_{n-1}' \arrow[r] & \cdots
\end{tikzcd}$$

The **homology** of a chain complex (C, ∂)

is $H_n(C) := \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$

Say: H_n measures how far (C, ∂) is from being exact.

Prop: A chain map

$$(C, \partial) \xrightarrow{f} (C', \partial')$$

induces a map of homology groups

$$H_n(C) \xrightarrow{f} H_n(C')$$

Pf sketch: $f\partial = \partial' f \Rightarrow f(\ker \partial) \subseteq \ker \partial'$

$f(\operatorname{Im} \partial) \subseteq \operatorname{Im} \partial'$.

Examples:

① $C_n = C_n^\Delta(X)$ simplicial chain complex

② $C_n = C_n(X)$ singular chain complex.

pf of thm: A continuous map $X \xrightarrow{f} Y$ induces a map $C_n(X) \xrightarrow{f_\#} C_n(Y)$

$$(\sigma : \Delta^n \to X) \longmapsto \left(\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[rr, bend right, "f_\# \sigma"'] & X \arrow[r, "f"] & Y \end{tikzcd} \right)$$

check: $f_\# \partial = \partial f_\#$

(exercise).

So $f_\#$ is a chain map!

prop $\Rightarrow$ $f_\# : C_n(X) \to C_n(Y)$

induces a map

$f_* : H_n(X) \to H_n(Y)$ $\square$

Rmk: ① $X \xrightarrow{g} Y \xrightarrow{f} Z$

we have $(fg)_* = f_* g_*$

② $X \xrightarrow{id} X$ induces identity map on $H_n(X) \to H_n(X)$.

Homotopy invariance.

Thm: If two maps $f, g : X \to Y$ are homotopic then they induce the same maps

$$f_* = g_* : H_n(X) \to H_n(Y) \qquad \forall n.$$

Pf: First we divide $\Delta^n \times I$ into simplices.

Let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$

$\Delta^n \times \{1\} = [w_0, \dots, w_n]$

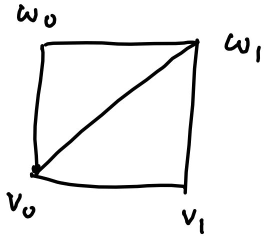
s.t. v_i, w_i have same image under projection $\Delta^n \times I \to \Delta^n$.

$\Delta^n \times I$ is a union of $(n+1)$-simplices of the form $[v_0, \ldots, v_i, w_i, \ldots, w_n]$

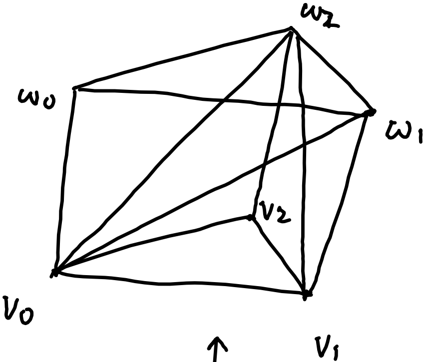
$i = 0, 1, \ldots, n.$

Examples:

$\Delta^1 \times I$:



$\Delta^2 \times I$:



3 3-simplices.

Consider a homotopy $F : X \times I \longrightarrow Y$

from f to g.

$\sigma : \Delta^n \longrightarrow X$ a singular simplex

Compose:

$$\begin{tikzcd}
\Delta^n \times I \arrow[r, "\sigma \times id"] \arrow[rr, bend right, "F \circ (\sigma \times id)"'] & X \times I \arrow[r, "F"] & Y
\end{tikzcd}$$

Define a map ("prism operator").

$$P : C_n(X) \longrightarrow C_{n+1}(Y)$$

$$P(\sigma) = \sum_i (-1)^i \, F \circ (\sigma \times id) \Big|_{[v_0 \ldots v_i w_i \ldots w_n]}$$

subsimplices of $\Delta^n \times I$.

claim: $\partial P = g_\# - f_\# - P\partial$

$$\begin{tikzcd}
& C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, "g_\#, f_\#"'] & C_{n-1}(X) \arrow[dl, "P"'] \arrow[d, "g_\#, f_\#"] \\
C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y)
\end{tikzcd}$$

intuition: $\partial P \sim$ boundary of $\Delta^n \times I$

$g_\# \sim \Delta^n \times \{1\}$ top

$f_\# \sim \Delta^n \times \{0\}$ bottom

$P\partial \sim \partial\Delta^n \times I$ sides

pf of claim:

$$\partial P(\sigma) = \sum_{j\le i} (-1)^i (-1)^j F\circ(\sigma\times id)\Big|_{[v_0 \dots \hat{v}_j \dots v_i, w_i \dots w_n]} + \sum_{j\ge i} (-1)^i (-1)^{j+1} F\circ(\sigma\times id)\Big|_{[v_0 \dots v_i, w_i \dots \hat{w}_j \dots w_n]}$$

sums of Terms with $i=j$ will cancel (exercise)

except $F\circ(\sigma\times id)\Big|_{[\underbrace{\hat{v}_0\ w_0\cdots w_n}_{\overset{\shortparallel}{\Delta^n\times\{1\}}}]} = g\circ\sigma = g_{\#}(\sigma)$.

and $-F\circ(\sigma\times id)\Big|_{[v_0\cdots v_n,\ \hat{w}_n]} = -f\circ\sigma = -f_{\#}(\sigma)$

Sums of terms with $i\neq j$ are

$$P\partial(\sigma) = \sum_{i<j}(-1)^i(-1)^j F\circ(\sigma\times id)\Big|_{[v_0\cdots v_i,\ w_i\cdots\hat{w}_j\cdots w_n]} + \sum_{i>j}(-1)^{i-1}(-1)^j F\circ(\sigma\times id)\Big|_{[v_0\cdots\hat{v}_j\cdots,\ v_i,\ w_i\cdots w_n]}$$

$$= -P\partial(\sigma).$$

claim is proven.

Finish proving thm:

It suffices to prove that $\forall \alpha \in C_n(X)$

if α is a cycle (i.e. $\partial\alpha = 0$)

then $g_\#(\alpha) - f_\#(\alpha)$ is a boundary (i.e. in $\operatorname{im}\partial$)

Indeed,

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\underset{0}{\underset{\shortparallel}{\partial\alpha)}} = \partial P(\alpha) \qquad \square.$$

Back to algebra:

Def: A chain homotopy between two chain maps $(C_\bullet, \partial) \overset{f_\#}{\underset{g_\#}{\rightrightarrows}} (C_\bullet', \partial)$

is a map $P : C_n \longrightarrow C_{n+1}$ s.t.

$$\partial P + P \partial = g_\# - f_\# .$$

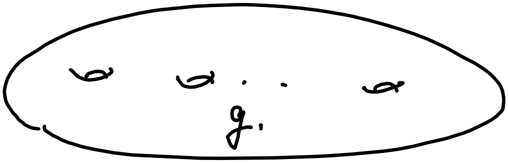
We have proven:

Prop: chain-homotopic maps induce the same homomorphism on homology.

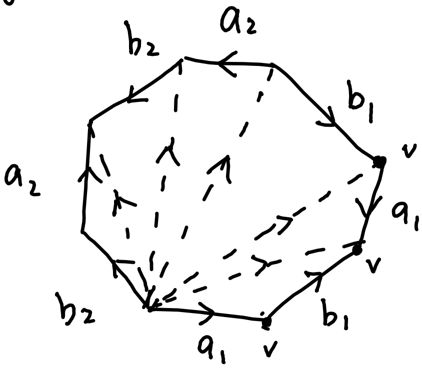
i.e. $f_* = g_* : H_n(C_\bullet) \to H_n(C_\bullet')$.

Example: Surfaces as Δ-complexes (HW).

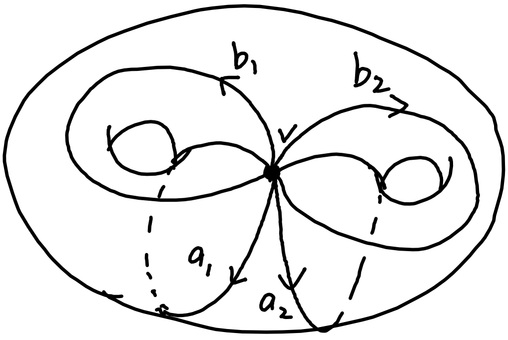
$\Sigma_g =$



$g = 2.$



$\rightsquigarrow$



4g-gon.

genus-g surface.

Last time :

We defined

$$C_n(X) = \mathbb{Z}\{\sigma \mid \sigma : \Delta^n \to X \text{ con't}\}$$

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

$$H_n(X) := \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$ singular homology group of X.

Today : Properties of $H_n(X)$.

Induced map.

Thm: A continuous map $f: X \to Y$ induces a homomorphism $f_*: H_n(X) \to H_n(Y)$ $\forall n$.

(Algebra)

$(C_\bullet, \partial)$

A chain complex is a sequence of abelian groups C_n, $n \in \mathbb{Z}$ with homomorphisms $\partial_n : C_n \to C_{n-1}$ s.t. $\partial_{n-1} \circ \partial_n = 0$.

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots$$

Suppose (C, ∂) and (C', ∂') are chain complexes.

A chain map $(C,\partial) \xrightarrow{f} (C',\partial')$ consists of maps $C_n \xrightarrow{f_n} C_n'$ s.t. $f\partial = \partial' f$

i.e. the diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_n' \arrow[r, "\partial_{n-1}"] & C_{n-1}' \arrow[r] & \cdots
\end{tikzcd}$$

The <u>homology</u> of a chain complex (C, ∂) is

$$H_n(C) := \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

<u>Say</u>: H_n measures how far (C, ∂) is from being exact.

<u>Prop</u>: A chain map

$$(C, \partial) \xrightarrow{f} (C', \partial')$$

induces a map of homology groups

$$H_n(C) \xrightarrow{f} H_n(C')$$

<u>Pf sketch</u>: $f\partial = \partial' f \Rightarrow f(\ker\partial) \subseteq \ker\partial'$

$f(\operatorname{Im}\partial) \subseteq \operatorname{Im}\partial'$!

Examples:

① $C_n = C_n^{\Delta}(X)$ simplicial chain complex

② $C_n = C_n(X)$ singular chain complex.

pf of thm: A continous map $X \xrightarrow{f} Y$ induces

a map $C_n(X) \xrightarrow{f_\#} C_n(Y)$

$$\left(\sigma : \Delta^n \to X\right) \longmapsto \left(\begin{tikzcd} \Delta^n \arrow[r, "\sigma"] \arrow[rr, bend right, "f_\# \sigma"'] & X \arrow[r, "f"] & Y \end{tikzcd}\right)$$

check: $f_\# \partial = \partial f_\#$

(exercise).

So $f_\#$ is a chain map!

prop $\Rightarrow$ $f_\# : C_n(X) \to C_n(Y)$

induces a map

$f_* : H_n(X) \to H_n(Y)$ □

Rmk: ① $X \xrightarrow{g} Y \xrightarrow{f} Z$

we have $(fg)_* = f_* g_*$

② $X \xrightarrow{id} X$ induces identity map on $H_n(X) \to H_n(X)$.

Say:

" The map $X \longmapsto H_n(X)$ is a functor from the category of top. spaces and cont. maps to the category of abelian groups. "

Homotopy invariance.

Thm: If two maps $f, g : X \to Y$ are homotopic then they induce the same maps

$$f_* = g_* : H_n(X) \to H_n(Y) \quad \forall n.$$

Pf: First we divide $\Delta^n \times I$ into simplices.

Let $\Delta^n \times \{0\} = [v_0, \cdots, v_n]$

$\Delta^n \times \{1\} = [w_0, \cdots, w_n]$

s.t. v_i, w_i have same image under projection $\Delta^n \times I \longrightarrow \Delta^n$.

$\Delta^n \times I$ is a union of $(n+1)$-simplices of the form $[v_0, \cdots, v_i, w_i, \cdots, w_n]$

$i = 0, 1, \cdots, n.$

Examples:

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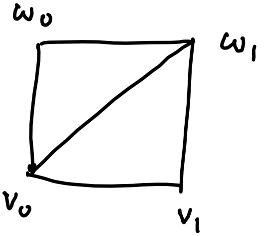
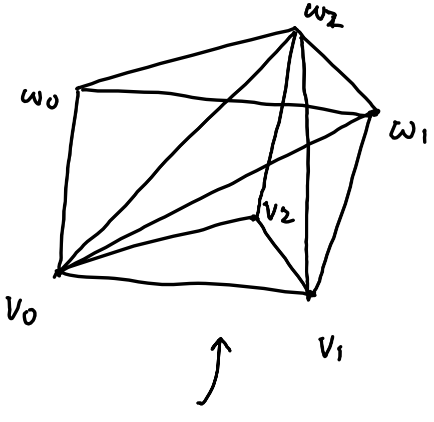


$\Delta^2 \times I$:



3 3-simplices.

Consider a homotopy $F : X \times I \longrightarrow Y$ from f to g.

$\sigma : \Delta^n \longrightarrow X$ a singular simplex

Compose:

$$\begin{tikzcd}
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\end{tikzcd}$$

Define a map ("prism operator").

$$P : C_n(X) \longrightarrow C_{n+1}(Y)$$

$$P(\sigma) = \sum_i (-1)^i F \circ (\sigma \times id) \Big|_{[v_0 \ldots v_i w_i \ldots w_n]}$$

subsimplices of $\Delta^n \times I$.

claim: $\partial P = g_\# - f_\# - P\partial$

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 & C_n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, "g_\#, f_\#"] & C_{n-1}(X) \arrow[dl, "P"'] \arrow[d, "g_\#, f_\#"] \\
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intuition: $\partial P \sim$ boundary of $\Delta^n \times I$

$g_\# \sim \Delta^n \times \{1\}$ top

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pf of claim:

$$\partial P(\sigma) = \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times id)\Big|_{[v_0 \dots \hat{v}_j \dots v_i, w_i \dots w_n]}$$

$$+ \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times id)\Big|_{[v_0 \dots v_i, w_i \dots \hat{w}_j \dots w_n]}$$

sums of Terms with $i=j$ will cancel (exercise)

except $F\circ(\sigma\times id)\Big|_{[\underbrace{\hat{v}_0 \ w_0\cdots w_n}_{\substack{\shortparallel \\ \Delta^n\times\{1\}.}}]} = g\circ\sigma = g_\#(\sigma)$

and $-F\circ(\sigma\times id)\Big|_{[v_0\cdots v_n, \hat{w}_n]} = -f\circ\sigma = -f_\#(\sigma)$

Sums of terms with $i\neq j$ are

$$P\partial(\sigma) = \sum_{i<j}(-1)^i(-1)^j F\circ(\sigma\times id)\Big|_{[v_0\cdots v_i, w_i\cdots\hat{w}_j\cdots w_n]}$$

$$+\sum_{i>j}(-1)^{i-1}(-1)^j F\circ(\sigma\times id)\Big|_{[v_0\cdots\hat{v}_j\cdots, v_i, w_i\cdots w_n]}$$

$$= -P\partial(\sigma).$$

claim is proven.

Finish proving thm:

It suffices to prove that $\forall\, \alpha \in C_n(X)$

if α is a cycle (i.e. $\partial\alpha = 0$)

then $g_\#(\alpha) - f_\#(\alpha)$ is a boundary (i.e. in $\operatorname{im}\partial$)

Indeed,

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\underset{0}{\underset{\shortparallel}{\partial(\alpha)}} = \partial P(\alpha)$$

□.

Back to algebra:

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is a map $P : C_n \longrightarrow C_{n+1}$ s.t.

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We have proven:

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i.e. $f_* = g_* : H_n(C_\bullet) \to H_n(C_\bullet')$.

Exact sequences

Goal: Find a relationship between the homology of X, A, and X/A.

Guess: $H_n(X/A) \overset{?}{=} \frac{H_n(X)}{H_n(A)}$ False, but close to be true.

Thm. If X is a space,
$A \subset X$ closed, nonempty, and is a deformation retract of a neighborhood in X,
then $\exists$ a long exact sequence:

$$\to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \cdots$$

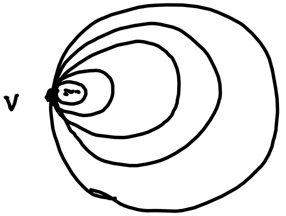
where $i : A \hookrightarrow X$

$j : X \twoheadrightarrow X/A$.

pf later.

If so, we say (X, A) is a good pair.

(nonexample:



).

Example: X = a Δ-complex (or CW complex).

A = a subcomplex

Cor: $\tilde{H}_n(S^n) = \mathbb{Z}$

$\tilde{H}_i(S^n) = 0 \quad \forall\, i \neq n.$

Pf. Induct on n.

$n=0$ ✓

$n>0$. Take $(X, A) = (D^n, S^{n-1})$

$X/A = D^n/S^{n-1} = S^n$.

$D^n \simeq * \Rightarrow \tilde{H}_i(D^n) = 0 \quad \forall\, i$

$\Rightarrow \tilde{H}_i(S^n) \xrightarrow[\cong]{\partial} \tilde{H}_{i-1}(S^{n-1})$

□.

Cor. (Brouwer's fixed pt theorem).

∂D^n is not a retract of D^n.

Hence every continuous $f: D^n \to D^n$ has a fixed pt.

Pf. Suppose not.

$$\begin{tikzcd}
\partial D^n \arrow[r, hook, "i"] \arrow[rr, bend right, "id"'] & D^n \arrow[r, two heads, "r"] & \partial D^n
\end{tikzcd}$$

apply $\tilde{H}_{n-1}$:

$$\begin{tikzcd}
\tilde{H}_{n-1}(\partial D^n) \arrow[r, "i_*"] \arrow[d, phantom, "\cong" rotate=90] & \tilde{H}_{n-1}(D^n) \arrow[r] \arrow[d, phantom, "\parallel"] & \tilde{H}_{n-1}(\partial D^n) \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} \arrow[r] \arrow[rr, bend right, "id"'] & 0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

contradiction.

$\square$.

① Long exact sequence of pairs

\+ ② excision

$\Rightarrow$ Thm.

① LES of a pair: $A \subseteq X$. subspace.

$$C_n(A) \longrightarrow C_n(X).$$

Define $C_n(X,A) := \dfrac{C_n(X)}{C_n(A)}.$

note: $\partial : C_n(X) \to C_{n-1}(X)$

induces $\bar{\partial} : C_n(X,A) \longrightarrow C_n(X,A).$ $\boxed{\text{STOP}}$

$\Rightarrow$ $(C_\cdot(X,A), \bar{\partial})$ is a chain complex

Define the <u>relative homology groups</u>

$$H_n(X,A) := H_n(C_\cdot(X,A), \bar{\partial}) = \frac{\operatorname{Ker} \bar{\partial}_n}{\operatorname{Im} \bar{\partial}_{n+1}}.$$

Elements in $H_n(X,A)$ are represented by <u>relative cycles</u>: $\alpha \in C_n(X)$ s.t. $\partial\alpha \in C_{n-1}(A)$.

α is trivial in $H_n(X,A)$ iff α is a <u>relative boundary</u>:

$$\alpha = \partial\beta + \gamma \quad , \quad \beta \in C_{n+1}(X), \ \gamma \in C_n(A).$$

<u>Note</u>: $A \subset X$ gives a SES of chain complexes:

$$0 \to C_\bullet(A) \to C_\bullet(X) \to \underbrace{\frac{C_\bullet(X)}{C_\bullet(A)}}_{=\, C_\bullet(X,A).} \to 0$$

$\boxed{\text{Algebra}}$

Lemma: A SES of chain complexes

$$0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$$

(i.e. i, j are chain maps).

induces a LES of homology groups

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

"connecting homomorphism".

Pf. sketch: key is to define $\partial : H_n(C) \to H_{n-1}(A)$.

$$\overset{\psi}{c} \longmapsto a.$$

$$\begin{tikzcd}
 & & A_{n-1} \arrow[d, "i"] & a \arrow[dd] \\
B_n \arrow[rr, "\partial"] \arrow[dd, "j"'] & & B_{n-1} & \\
 & b \arrow[r, mapsto] \arrow[d] & \partial b & = i(a) \\
C_n & c & &
\end{tikzcd}$$

since $j(\partial b) = \partial(j(b)) = \partial c = 0$.

exactness.

$\partial(a) = 0$
since
$\partial(i(a)) = \partial\partial b = 0$

check ∂ satisfies properties needed (exercise) $\uparrow$ $\square$.

"diagram chasing"

<u>Back to topology</u>.

<u>prop</u>. If $A \overset{i}{\subset} X$

then we have a LES

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

("LES of a pair").

<u>prop</u>.

1. A map $f: \begin{matrix} X & \to & Y \\ \cup\! | & & \cup\! | \\ A & & B \end{matrix}$ s.t. $f(A) \subset B$

   induces $f_* : H_n(X,A) \to H_n(Y,B)$.
2. If $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs, then $f_* = g_*$.

<u>prop</u>. (LES of a triple)

If $B \subseteq A \subseteq X$

then $\exists$ a LES of the triple (X, A, B)

$$\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

<u>Pf</u>: SES of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & C_n(A,B) \arrow[r] \arrow[d, equal] & C_n(X,B) \arrow[r] \arrow[d, equal] & C_n(X,A) \arrow[r] \arrow[d, equal] & 0. \\
& \frac{C_n(A)}{C_n(B)} & \frac{C_n(X)}{C_n(B)} & \frac{C_n(X)}{C_n(A)} &
\end{tikzcd}$$

Example: $X = [0,1]$

$A = \partial X$

$H_1(X) = 0$ since X contractible

However, $H_1(X, A) \cong \mathbb{Z}$.

$$\begin{tikzcd}
H_1(X) \arrow[r] \arrow[d, equal] & H_1(X,A) \arrow[r, "\partial"] & H_0(A) \arrow[r, "i"] \arrow[d, equal] & H_0(X) \arrow[d, equal] \\
0 & & \mathbb{Z}\{u, w\} \arrow[r] & \mathbb{Z}\{u\}
\end{tikzcd}$$

$u \longmapsto u$

$w \longmapsto u$ since $w - u = \underset{=0}{\partial \ell}$

$H_1(X,A) \cong \ker i \cong \mathbb{Z}$

A representative of a generator of $H_1(X,A)$ is

$\ell : [v_0, v_1] \longrightarrow [u, w]$ in $C_1(X)$

Note: $\partial \ell = w - u \neq 0$ so ℓ is not a 1-cycle

However, $\partial \ell = w - u \in C_1(A)$

so $\partial \ell$ is a relative 1-cycle in $C_1(X, A)$.

The connecting homomorphism is

$$
\begin{aligned}
H_1(X, A) &\xrightarrow{\partial} H_0(A) \\
\ell &\longmapsto \partial \ell = w - u \in \operatorname{Ker} i
\end{aligned}
$$

(II) Excision.

Thm. (Excision).

Given $Z \subseteq A \subseteq X$ s.t. $\overline{Z} \subseteq \mathring{A}$,

the inclusion

$$(X - Z, A - Z) \hookrightarrow (X, A)$$

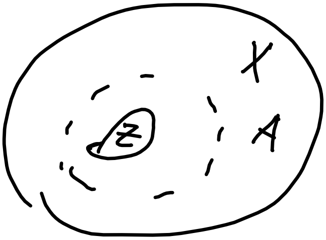
induces an isomorphism

$$H_n(X - Z, A - Z) \longrightarrow H_n(X, A) \quad \forall n.$$

Equivalently,

given $A, B \subseteq X$ s.t.

$$\mathring{A} \cup \mathring{B} = X,$$



the inclusion

$$(B, A \cap B) \hookrightarrow (X, A)$$

induces an iso

$$H_n(B, A \cap B) \longrightarrow H_n(X, A).$$

Take $B = X - Z$.

<u>pf sketch</u>: We will prove the second version

Define

$$C_n(A+B) := \mathbb{Z}\left\{ \sigma : \Delta^n \to X \;\middle|\; \begin{array}{l} \mathrm{im}(\sigma) \subseteq A \\ \text{or } \mathrm{im}(\sigma) \subseteq B \end{array} \right\}.$$

$$\subseteq C_n(X).$$

<u>claim</u>: $C_n(A+B) \overset{\iota}{\hookrightarrow} C_n(X)$

is a chain homotopy equivalent.

i.e. $\exists \rho$ s.t. $\rho \circ \iota$ and $\iota \circ \rho$ are chain homotopic to identity.

<u>idea</u>: barycentric subdivide: $\Delta^n \xrightarrow{\sigma} X = A \cup B$.

consider $\sigma^{-1}(A)$ and $\sigma^{-1}(B)$

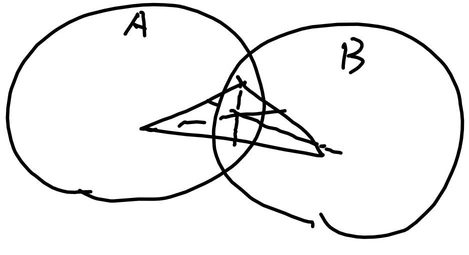
ρ divides Δ into smaller pieces

s.t. each piece is

either in A or in B.

(Lebesgue's number lemma

$\Rightarrow$ this is possible).



$$C_n(A) \subseteq C_n(A+B) \underset{\uparrow}{\subseteq} C_n(X)$$

chain h.e.

① $$\frac{C_n(A+B)}{C_n(A)} \xhookrightarrow{\iota} \frac{C_n(X)}{C_n(A)}$$ induces iso on $H_\bullet$

② $$\begin{array}{ccc} \dfrac{C_n(B)}{C_n(A\cap B)} & \longrightarrow & \dfrac{C_n(A+B)}{C_n(A)} \\ \shortparallel & & \shortparallel \\ \mathbb{Z}\{\sigma:\Delta^n\to B \mid \cancel{\sigma\cap A=\phi}\} & & \mathbb{Z}\{\sigma:\Delta^n\to B \mid \cancel{\sigma\cap A=\phi}\} \\ \mathrm{im}(\sigma)\not\subseteq A & & \mathrm{im}(\sigma)\not\subseteq A \end{array}$$

is an iso. of chain complexes.

① + ② $\Rightarrow$ $$H_n(B, A\cap B) \cong H_n(X, A)$$

□.

Cor: For a good pair (X, A), the quotient map $q : (X,A) \to (X/A, A/A)$ induces an isomorphism

$$\begin{tikzcd}
q_* : H_n(X,A) \arrow[r, "\sim"] & H_n(X/A, *) \arrow[d, phantom, sloped, "\cong"] \\
 & \tilde{H}_n(X/A).
\end{tikzcd}$$

Rmk: Hence, we have a LES

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & H_{n-1}(A) \arrow[r] & \cdots \\
 & & & \tilde{H}_n(X/A) & &
\end{tikzcd}$$

(This proves the theorem we stated last time).

pf (Thm ⇒ Cor):

Good pair: $A \subseteq V \subseteq X$

V open, $\overline{A} \subseteq V$, $V \rightsquigarrow A$.

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "①"] \arrow[d, "q_*"'] & H_n(X, V) \arrow[d, "q_*"] & H_n(X-A, V-A) \arrow[l, "③"'] \arrow[d, "⑤"', "q_*"] \\
H_n(X/A, *) \arrow[r, "②"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "④"']
\end{tikzcd}$$

① is iso: In LES of triple (X, V, A),
$H_n(V, A) = 0$ since $(V, A) \simeq (A, A)$
as pairs.

② is iso: Same argument on $(X/A, V/A, A/A)$.

③, ④ are iso by excision.

⑤ is iso since q is an ~~isomorphism~~ homeomorphism away from A.

$\boxed{\text{STOP}}$. $\square$.

Cor:

$$\tilde{H}_n\Big(\bigvee_\alpha X_\alpha\Big) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$$

provided that $(X_\alpha, \underset{\substack{\uparrow \\ \text{base point.}}}{x_\alpha})$ is good ∀ α.

recall:

$$\bigvee_\alpha X_\alpha = \left. \coprod X_\alpha \middle/ {x_\alpha \sim x_\beta} \right. .$$

Pf:

$$\begin{array}{ccc} H_n\Big(\coprod_\alpha X_\alpha, \coprod_\alpha \{x_\alpha\}\Big) & \cong & \tilde{H}_n\Big(\bigvee_\alpha X_\alpha\Big) \\ \wr\!\wr & & \\ \bigoplus_\alpha H_n(X_\alpha, \{x_\alpha\}) & \cong & \bigoplus_\alpha \tilde{H}_n(X_\alpha). \end{array}$$

□.

Last time:
- LES of pair
- Excision.

Today: The LES of a pair is natural.

Prop: Given $f:(X,A)\longrightarrow(Y,B)$ a map of pairs. the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d, "f_*"] & H_n(X) \arrow[r] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r] & H_n(Y) \arrow[r] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

Say: ∂ is a natural transformation from the functor $H_n(-,-)$ to the functor $H_{n-1}(-)$.

Pf: Just algebra:

prop. (algebra):

If

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r] \arrow[d, "f_\#"] & B_\bullet \arrow[r] \arrow[d, "f_\#"] & C_\bullet \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & A_\bullet' \arrow[r] & B_\bullet' \arrow[r] & C_\bullet' \arrow[r] & 0
\end{tikzcd}$$

commutes as chain maps.

then

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r] \arrow[d] & H_n(B_\bullet) \arrow[r] \arrow[d] & H_n(C_\bullet) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d] & \cdots \\
\arrow[r] & H_n(A') \arrow[r] & H_n(B_\bullet') \arrow[r] & H_n(C_\bullet') \arrow[r, "\partial"] & H_{n-1}(A') \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

pf by diagram chasing.

Rmk: Similar naturality result holds for

- LES of triples
- LES of $\tilde{H}(A)$, $\tilde{H}(X)$ and $\tilde{H}(X/A)$.

  for good pairs.

Equivalence of simplicial and singular homology.

Suppose X is a Δ-complex:

$\exists$ a natural chain map

$$C_n^{\Delta}(X) \hookrightarrow C_n(X).$$

$$\rightsquigarrow \quad H_n^{\Delta}(X) \longrightarrow H_n(X).$$

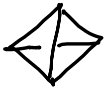
Thm $\quad H_n^{\Delta}(X) \cong H_n(X).$

Pf: First consider when X is finite-dimensional

i.e. X has no simplex of dim $> d$.

Let $X^k := \bigcup$ simplices of dimension $\leq k$.

"k-skeleton".

e.g. $X = \Delta^3$, $k = 1$, $(\Delta^3)^1 =$ 

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^d = X.$$

Consider the LES of pair (X^k, X^{k-1}).

$$\begin{tikzcd}
H^{\Delta}_{n+1}(X^k, X^{k-1}) \arrow[r, "\partial"] \arrow[d, "①"] & H^{\Delta}_n(X^{k-1}) \arrow[r] \arrow[d, "②"] & H^{\Delta}_n(X^k) \arrow[r] \arrow[d, "③"] & H^{\Delta}_n(X^k, X^{k-1}) \arrow[r, "\partial"] \arrow[d, "④"] & H^{\Delta}_{n-1}(X^{k-1}) \arrow[d, "⑤"] \\
H_{n+1}(X^k, X^{k-1}) \arrow[r, "\partial"] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1})
\end{tikzcd}$$

commutes as before.

We induct on k.

<u>Goal</u>: ③ is an iso. $\forall n$.

We know: ② and ⑤ are iso by induction hypothesis.

We claim that ① and ④ are iso $\forall n$.

Note: $C_n^{\Delta}(X^k, X^{k-1}) = 0 \quad \forall\, n \neq k.$

$$C_k^{\Delta}(X^k, X^{k-1}) = \mathbb{Z}\{k\text{-simplices of } X\}.$$

$$\Rightarrow \quad H_n^{\Delta}(X^k, X^{k-1}) = C_n^{\Delta}(X^k, X^{k-1}) = \begin{cases} 0 & n \neq k \\ \mathbb{Z}\{k\text{-simplices of } X\} & n = k. \end{cases}$$

Consider singular homology.

(X^k, X^{k-1}) a good pair

(last time).

$$\Rightarrow \quad H_n(X^k, X^{k-1}) \cong H_n(X^k/X^{k-1}, *) \cong \tilde{H}_n(X^k/X^{k-1}).$$

$$\frac{X^k}{X^{k-1}} \cong \frac{\coprod_\alpha \Delta_\alpha^k}{\coprod_\alpha \partial\Delta_\alpha^k} \cong \bigvee_\alpha \underbrace{\left(\Delta_\alpha^k/\partial\Delta_\alpha^k\right)}_{\hookrightarrow\, \cong S^k}.$$

$$\tilde{H}_n(X^k/X^{k-1}) \underset{(HW)}{\cong} \bigoplus_\alpha \tilde{H}_n(\Delta_\alpha^k/\partial\Delta_\alpha^k)$$

$$= \begin{cases} 0 & n \neq k \\ \mathbb{Z}\{k\text{-simples}\} & n = k. \end{cases}$$

so ① . ④ are iso $\forall\, n$.

Algebra

Lemma (Five lemma).

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "①"'] & B \arrow[r] \arrow[d, "②"'] & C \arrow[r] \arrow[d, "③"'] & D \arrow[r] \arrow[d, "④"'] & E \arrow[d, "⑤"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E
\end{tikzcd}$$

If two rows are exact sequences of ab. gr.
and ① ② ④ ⑤ are iso.
then ③ is an iso.

Pf by diagram chasing.

Lemma ⇒ ③ is iso.

Now consider when X is ∞-dimensional.

$X^0 \subseteq X^1 \subseteq \cdots \subseteq X^n \subseteq \cdots \qquad X = \bigcup_{k=0}^{\infty} X^k$

Key: A compact subset $C \subseteq X$ can meet only finitely many simplices, hence $C \subseteq X^k$ for some k.

Consider $H_n^\Delta(X) \longrightarrow H_n(X)$.

surjectivity: take $[z] \in H_n(X)$
represented by $z \in C_n(X)$

$z = \sum_\alpha n_\alpha \sigma_\alpha$ finite sum.

so image in X^k for some k.

Before: $H_n^\Delta(X^k) \xrightarrow{\cong} H_n(X^k)$.

So z is homologous in X^k (hence in X) to a simplicial cycle.

injectivity: take $[z]$ in the kernel

so $z \in C_n^\Delta(X)$ and $z = \partial y$, $y \in C_{n+1}(X)$.

Same argument, y is homologous in X to a simplicial cycle. $\square$

Cor: Suppose X is a Δ-complex

$A \subseteq X$ a subcomplex.

Then $H_n^{\Delta}(X, A) \cong H_n(X, A)$.

Pf: LES of pairs. + five lemma. (exercise).

Say: Before: homology theory. (Hatcher 2.1).

Now: Computations.

(Hatcher 2.2).

Degree

Consider a map $f: S^n \to S^n$. $(S^n \subseteq \mathbb{R}^{n+1})$.

$H_n(S^n) \cong \mathbb{Z}$.

We fix a generator $1 \in \mathbb{Z}$.

$$\begin{tikzcd}
f_* : H_n(S^n) \arrow[r] \arrow[d, phantom, "\cong"] & H_n(S^n) \arrow[d, phantom, "\cong"] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \\
1 \arrow[r, mapsto] & d = f_*(1).
\end{tikzcd}$$

d is called the degree of f or $\deg(f)$.

Basic properties:

(a) $\deg(id) = 1$.

(b) $\deg(f) = 0$ if f is not onto.

pick $x \notin Im(f)$.

then

$$\begin{tikzcd}
 & S^n \setminus \{x\} \cong \mathbb{R}^n \simeq * . \arrow[d, hook] \\
S^n \arrow[ur] \arrow[r, "f"'] & S^n
\end{tikzcd}$$

(c) $f \simeq g \Rightarrow \deg f = \deg g$.

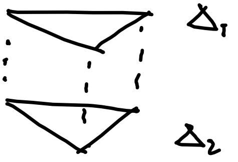
Thm (Brouwer) : Converse is true.

(d) $\deg(fg) = \deg f \cdot \deg g$.

Hence if f is a homeo. (or homotopy. equiv.)
then $\deg f = \pm 1$.

(e) If $f: S^n \to S^n$ is a reflection
e.g. $f(x_0, \cdots, x_n) = (-x_0, x_1, \cdots, x_n)$.
then $\deg f = -1$.

Pf: S^n as a Δ-complex. with two n-simplices as two hemispheres.
a generator of $H_n(S^n)$ is $\Delta_1^n - \Delta_2^n$
reflection $\Delta_1^n - \Delta_2^n \longmapsto \Delta_2^n - \Delta_1^n$



(f) The antipodal map

$$-id : S^n \longrightarrow S^n$$

$$(x_0, \cdots, x_n) \longmapsto (-x_0, \cdots, -x_n)$$

has degree $(-1)^{n+1}$

since it is a composition of $(n+1)$ reflections.

(g) If $f: S^n \to S^n$ has no fixed point

i.e. $\nexists\, x \in S^n$ s.t. $x = f(x)$.

then $\deg f = (-1)^{n+1}$.

Pf: If so, then

$$f_t(x) := \frac{(1-t)f(x) - tx}{\| (1-t)f(x) - tx \|} \in S^n$$

→ straight line from $f(x)$ to $-x$. avoids 0.

f_t is a homotopy from f to $-id$.

$f \simeq -id \Rightarrow \deg(f) = \deg(-id) = (-1)^{n+1}$

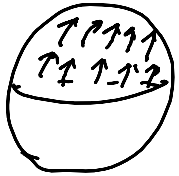
$\square$.

Application:

Thm (Hairy ball thm)

S^n has a nonvanishing vector field iff n is odd.

$\boxed{\text{STOP}}$



Pf: $(\Leftarrow)$. $n = 2k-1$.

$$v(x_1, x_2, \dots, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$$

check: $v(\vec{x}) \perp \vec{x}$. tangent vector.

$(\Rightarrow)$ Suppose $x \mapsto v(x)$ is a vector field.

s.t. $v(x) \perp x$ and $v(x) \neq 0$

normalize s.t. $\|v(x)\| = 1 \quad \forall x$.

Take $f_t(x) := (\cos t)\, x + (\sin t)\, v(x)$

$t \in [0, \pi]$.

$\|f_t(x)\| = 1 \quad \forall x$.

$f_t : S^n \to S^n$, $f_0 = id$, $f_\pi = -id$.

so $id \simeq -id \Rightarrow \deg(id) = \deg(-id)$

$\Rightarrow \quad 1 = (-1)^{n+1} \quad \Rightarrow \quad n$ odd. $\square$.

Related questions:

Q: How many independent vector fields are there on S^n?

Adams (1962): homotopy ~~theory~~ K-theory.

Q: When does a manifold M^n have a nonvanishing vector field?

Iff $\chi(M^n) = 0$. (characteristic classes).

Thm. $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.

(i.e. $\forall \underset{\neq 1}{g} \in G$, $\forall x \in S^n$. $g \cdot x \neq x$).

Pf. Suppose $G \curvearrowright S^n$.

then $G \longrightarrow \mathrm{Homeo}(S^n) \to \mathrm{Iso}(H_n(S^n)) = \{\pm I\}$

Free action $\Rightarrow$ $\forall \underset{\neq 1}{g} \in G$, $\deg(g) = (-1)^{n+1}$

$= -1$ n even

so $G \xrightarrow{\deg} \{\pm 1\}$ has trivial kernel.

$\Rightarrow G \cong \mathbb{Z}/2\mathbb{Z}$.

$\square$.

Computing deg f using local data.

Suppose $f: S^n \to S^n$ $(n>0)$ has the property that $\exists\, y \in S^n$ s.t. $f^{-1}(y)$ is finite.

$$f^{-1}(y) = \{x_1, \dots, x_m\}$$

Let $U_1 \dots U_m$ be disjoint nbhd of $x_1 \dots x_m$ s.t. $f(U_i) \subseteq V$ a nbhd of y.

so $f(U_i - x_i) \subset V - y$.

We have:

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[ld, "\cong"', "\text{excision}"] \arrow[d] & H_n(V, V-y) \arrow[d, "\cong"] \\
H_n(S^n, S^n - x_i) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[lu, "\cong"] \arrow[u] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

LES of pairs

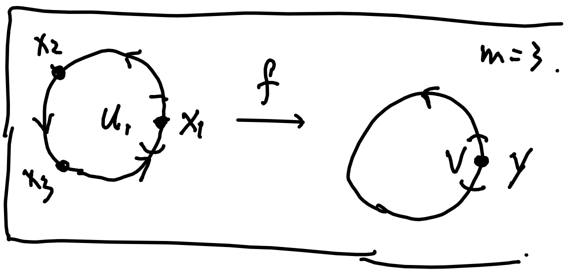
So the top f_* is $\mathbb{Z} \to \mathbb{Z}$.

f_* is multiplication by an integer called the <u>local degree</u> of f at x_i,

$\deg f\big|_{x_i}$.

<u>Example</u>: $n=1$.

$$f: S^1 \longrightarrow S^1 \qquad S^1 \subseteq \mathbb{C}.$$
$$z \longmapsto z^m$$



$$\begin{tikzcd} f_*: H_1(S^1) \arrow[r] & H_1(S^1) \\ \mathbb{Z} \arrow[u, phantom, "\cong" description] \arrow[r] & \mathbb{Z} \arrow[u, phantom, "\cong" description] \\ 1 \arrow[r, mapsto] & m. \end{tikzcd}$$

$\deg f = 3$.

f is a covering map $\Rightarrow$ f is locally a homeo

i.e. $f\big|_{U_i} : U_i \longrightarrow V_i$ is a homeo.

so $\deg f\big|_{x_i} = 1 \quad \forall i = 1, 2, \cdots, m$

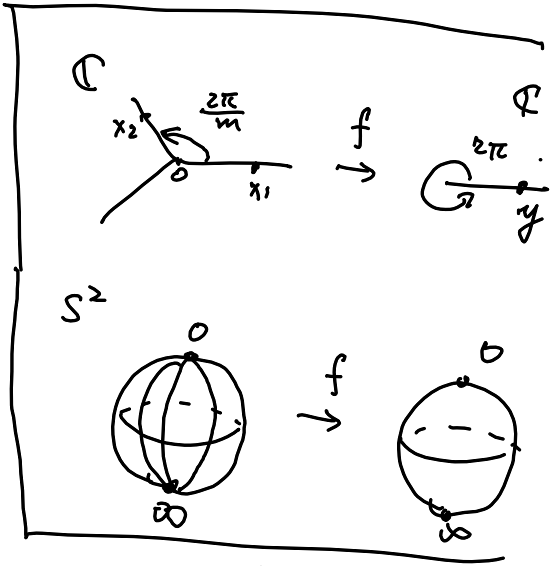
prop: $\deg f = \sum_i \deg f|_{x_i}$.

Pf: (HW exercise).

Example: Consider

$$\begin{tikzcd} f: S^2 \arrow[r] \arrow[d, equal] & S^2 \arrow[d, equal] \\ \mathbb{C}\cup\{\infty\} \arrow[r] & \mathbb{C}\cup\{\infty\}. \\ z \arrow[r, mapsto] & z^m \end{tikzcd}$$

Compute $\deg f$ in two ways.



① $y = 1 \in \hat{\mathbb{C}}$

$f^{-1}(y) = \{x \mid x^m = 1\}$.

note f is a homeomorphism near each $x \in f^{-1}(y)$

in fact, $f: \mathbb{C}^\times \to \mathbb{C}^\times$ is a covering map.

so $\deg f|_{x_i} = 1 \quad \forall i = 1, \dots, m$.

prop $\Rightarrow$ $\deg f = \sum_{i=1}^{m} \deg f|_{x_i} = m$,

② $y = 0 \in \hat{\mathbb{C}}$. , $f^{-1}(y) = \{0\}$.

$\deg f|_0 = m = \deg f$.

intuition: $f : S^2 \to S^2$: $\deg f|_{x_i} = d$

means near x_i, $f : U_i \to V$

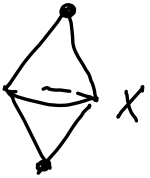
looks like the map $z \mapsto z^m$.

Prop: For $f: S^n \to S^n$

consider $Sf: \underset{\substack{\cong \\ S^{n+1}}}{S(S^n)} \longrightarrow \underset{\substack{\cong \\ S^{n+1}}}{\overset{\nearrow \text{ suspension}}{S(S^n)}}$.

Then $\deg(Sf) = \deg(f)$.

Recall: $S(X) = C(X) \big/ \text{base}$



Pf: Your HW: $$\underset{\substack{\circlearrowleft \\ f_*}}{\tilde{H}_n(X)} \cong \underset{\substack{\circlearrowleft \\ Sf_*}}{\tilde{H}_{n+1}(SX)}.$$ □.

Cor: For any $m \in \mathbb{Z}$,

$\exists\ f: S^n \to S^n$ with $\deg(f) = m$.

CW complex

Def: A CW complex is a space X obtained in the following way:

(1) Start with a discrete set X^0.

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e_α^n via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$.

i.e. $X^n = X^{n-1} \coprod_\alpha D_\alpha^n \Big/ \sim$

by $x \sim \varphi_\alpha(x) \quad \forall x \in \partial D_\alpha^n = S^{n-1}$

so as a set, $X^n = X^{n-1} \coprod_\alpha \mathring{D}_\alpha^n$

(3) Either $X = \bigcup_{n=0}^{d} X^n$ (finite-dim)

or $X = \bigcup_{n=0}^{\infty} X^n$ (∞-dim).

with "weak topology"

i.e. $A \subseteq X$ is open iff $A \cap X^n$ is open $\forall n$.

<u>Rmk</u>. (Δ-complexes v.s. CW complexes).

Δ-complex = $\coprod$ simplices $\Big/$ linear maps

CW-complex = $\coprod$ closed disks $\Big/$ continuous maps.

$\{\Delta\text{-complexes}\} \subset \{\text{CW-complexes}\}$.

Examples.

① $X = S^1$.

$X^0 = \bullet$

$X^1 = X^0 \amalg D^1 \big/ \varphi$ = $\bullet \cup$ 

$\varphi : S^0 \to X^1$

$0 \longmapsto *$

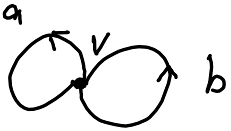
$1 \longmapsto *$.

② $X = X^1$ is a graph: 

③ $X = T^2$.

$X^0 = \overset{v}{\bullet}$

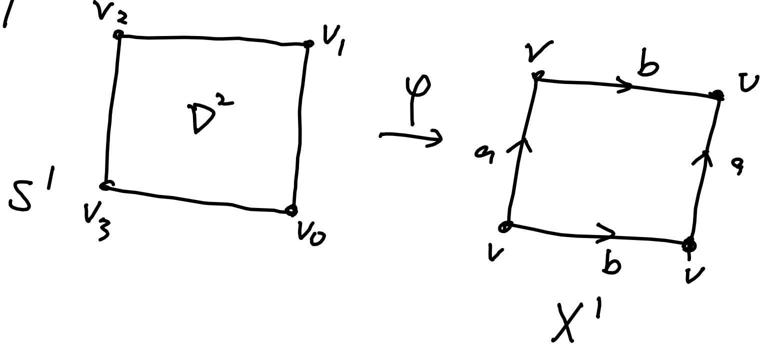
$X^1 =$



$X^2 = X^1 \cup D^2 / \varphi$

where $\varphi : \partial D^2 = S^1 \longrightarrow X^1 =$  .

by :



Similarly by $X = \Sigma_g$.

4g-gon $\longrightarrow \Sigma_g$.

④. $X = S^n$ with one 0-cell
and one n-cell.

$$X^0 = \{e^0\} = X^1 = X^2 = \cdots = X^{n-1}$$

with attaching map $\varphi : \underset{\substack{\shortparallel \\ \partial D^n}}{S^{n-1}} \longrightarrow X^{n-1} = \{*\}$.

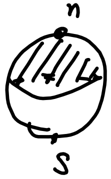
so $X^n = D^n \big/ {\partial D^n \sim *} = S^n$.

⑤ $X = \mathbb{R}P^n$ real projective space

$= \{ \text{1-d subspaces of } \mathbb{R}^{n+1} \}$.

$= \mathbb{R}^{n+1} - 0 \big/ {v \sim \lambda v,\ \lambda \in \mathbb{R}^\times}$.

$\cong S^n \big/ {v \sim -v}$.



$\cong D^n \big/ {v \sim -v}$
for $v \in \partial D^n$.

Note: $\overset{= S^{n-1}}{\partial D^n} / _{v \sim -v} = \mathbb{R}P^{n-1}$.

so $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup D^n$ a single n-cell

attaching map $\varphi : \partial D^n = S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$
$v \longmapsto [v]$

think

$$\mathbb{R}P^n = \{ [x_0 : x_1 : \cdots : x_n] / x_i \in \mathbb{R} \}.$$

$$\begin{array}{ccc} & \shortparallel & \\ \{ [x_0 : x_1 \cdots : x_{n-1} : 0] \} & \cup & \{ [x_0 : \cdots : x_{n-1} : 1] \} \\ \shortparallel & & \cong \\ \mathbb{R}P^{n-1} & \cup & \mathbb{R}^n \end{array}$$

Inductively,

$$\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$$

1-cell in each dimension.

$$\mathbb{R}P^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n \cup \cdots$$

⑥ $\mathbb{C}P^n = \{ \text{1-dim}_{\mathbb{C}} \text{ subspaces in } \mathbb{C}^{n+1} \}$.

$$= \mathbb{C}^{n+1} - 0 \Big/ \mathbb{C}^{\times}$$

Exercise:

$$\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

properties of CW complexes.

(1) CW complexes are Hausdorff, locally contractible, paracompact.

Finite CW complexes are compact.

(2) Every compact subspace of a CW complex is contained in a finite subcomplex.

(3) (X, A) is a good pair.

$$\underset{\text{CW}}{\downarrow}\qquad \underset{\text{subcomplex}}{\downarrow}$$

Most spaces (manifolds, varieties , etc).
are CW complexes .

Cellular homology.

Lemma: For X a CW complex.

(a) $$H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \mathbb{Z}\{\text{n-cells of } X\}. & k=n \end{cases}$$

(b) $H_k(X^n)=0 \quad \forall k>n.$

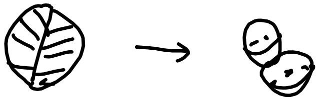
(c) $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is an iso $\forall k<n$
and surjective for $k=n$.

证 (a) (X^n, X^{n-1}) good pair.

$$\tilde{H}_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1})$$

$$= X^{n-1} \coprod_\alpha D_\alpha^n \Big/ X^{n-1} = \coprod_\alpha D_\alpha^n \Big/ \underset{\cong X^{n-1}}{\coprod_\alpha \partial D_\alpha^n}$$

$$= \bigvee_\alpha S^n.$$



$X^n \longrightarrow X^n / X^{n-1}$

(b) LES of (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \xrightarrow{i_*} H_k(X^n) \to H_k(X^n, X^{n-1})$$

$k \neq n \Rightarrow i_*$ is onto

$k \neq n-1 \Rightarrow i_*$ is 1-1.

So

$$\underbrace{H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1})}_{\text{iso.}} \to \underbrace{\underset{\text{1-1}}{\underset{\downarrow}{H_k(X^k)}} \to \underset{\text{onto.}}{\underset{\downarrow}{H_k(X^{k+1})}}}$$

$\Downarrow$

(b) since $H_k(X^0) = 0 \quad \forall k > 0$.

If X finite-dim., (c) ✓

If X is ∞-dim., to show surjectivity.

Take $[z] \in H_n(X)$. , z a representative

$z = \sum_\alpha n_\alpha \sigma_\alpha$ $\qquad$ σ_α has compact image

so contained in X^N.

then proceed. ...

□.

CW complex

Def: A CW complex is a space X obtained in the following way:

(1) Start with a discrete set X^0.

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e_α^n via maps $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$.

i.e. $X^n = X^{n-1} \coprod_\alpha D_\alpha^n \big/ \sim$

by $x \sim \varphi_\alpha(x) \quad \forall x \in \partial D_\alpha^n = S^{n-1}$

so as a set, $X^n = X^{n-1} \coprod_\alpha \mathring{D}_\alpha^n$

(3) Either $X = \bigcup_{n=0}^{d} X^n$ (finite-dim)

or $X = \bigcup_{n=0}^{\infty} X^n$. ($\infty$-dim).

with "weak topology"

i.e. $A \subseteq X$ is open iff $A \cap X^n$ is open $\forall n$.

<u>Rmk</u>. (Δ-complexes v.s. CW complexes).

Δ-complex $= \coprod$ simplices. / linear maps

CW-complex $= \coprod$ closed disks / continuous maps.

$\{\Delta\text{-complexes}\} \subset \{\text{CW-complexes}\}$.

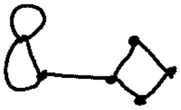
Examples.

① $X = S^1$.

$X^0 = \bullet$

$X^1 = X^0 \amalg D^1 / \varphi = \bullet \cup$ 

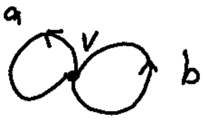
$$\begin{array}{l} \varphi: S^0 \to X^1 \\ \quad 0 \longmapsto * \\ \quad 1 \longmapsto * . \end{array}$$

② $X = X^1$ is a graph: 

③ $X = T^2$.

$X^0 = \overset{v}{\bullet}$

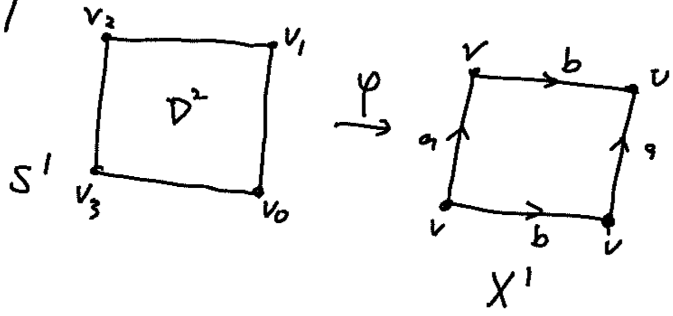
$X^1 =$



$X^2 = X^1 \cup D^2 / \varphi$

where $\varphi : \partial D^2 = S^1 \longrightarrow X^1 =$ .

by :



Similarly by $X = \Sigma_g$.

$4g$-gon $\longrightarrow \Sigma_g$.

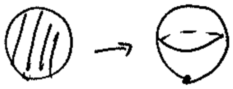
$\boxed{\text{STOP}}$

④. $X = S^n$ with one 0-cell
and one n-cell.

$$X^0 = \{e^0\} = X^1 = X^2 = \cdots = X^{n-1}$$

with attaching map $\varphi : \underset{\underset{\partial D^n}{\shortparallel}}{S^{n-1}} \longrightarrow X^{n-1} = \{*\}$.

so $X^n = D^n \big/ \partial D^n \sim * = S^n$.



⑤ $X = \mathbb{R}P^n$ real projective space

$= \{$ 1-d subspaces of $\mathbb{R}^{n+1} \}$.

$= \mathbb{R}^{n+1} - 0 \big/ v \sim \lambda v,\ \lambda \in \mathbb{R}^{\times}$.

$\cong S^n \big/ v \sim -v$.

$\cong D^n \big/ v \sim -v$ for $v \in \partial D^n$.

Note: $\overset{= S^{n-1}}{\partial D^n} \big/ _{v \sim -v} = \mathbb{R}P^{n-1}$.

so $\quad \mathbb{R}P^n = \mathbb{R}P^{n-1} \cup D^n \quad$ a single n-cell

attaching map $\varphi : \partial D^n = S^{n-1} \longrightarrow \mathbb{R}P^{n-1}$

$$v \longmapsto [v]$$

think

$$\mathbb{R}P^n = \{ [x_0 : x_1 : \cdots : x_n] \,/\, x_i \in \mathbb{R} \} .$$

$$\begin{array}{ccc} & \| & \\ \{ [x_0 : x_1 \cdots : x_{n-1} : 0] \} & \cup & \{ [x_0 : \cdots : x_{n-1} : 1] \} \\ \| & & \cong \\ \mathbb{R}P^{n-1} & \cup & \mathbb{R}^n \end{array}$$

Inductively,

$$\mathbb{R}P^n = e^0 \cup e^1 \cup e^2 \cup \cdots \cup e^n$$

1-cell in each dimension.

$$\mathbb{R}P^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n \cup \cdots$$

⑥ $\mathbb{C}P^n = \{ 1\text{-dim}_{\mathbb{C}} \text{ subspaces in } \mathbb{C}^{n+1} \}$.

$$= \mathbb{C}^{n+1}-\{0\} \Big/ \mathbb{C}^{\times}$$

So

$$\mathbb{C}P^n = \frac{\mathbb{C}^{n+1}-\phi}{\mathbb{C}-\phi} = \mathbb{C}^n + \mathbb{C}^{n-1} + \cdots + \mathbb{C} + 1.$$

Exercise:

$$\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

properties of CW complexes.

(1) CW complexes are Hausdorff, Locally contractible, paracompact (so has partition of 1).

Finite CW complexes are compact.

(2) Every compact subspace of a CW complex is contained in a finite subcomplex.

(3) $(\underset{\text{CW}}{\underset{\downarrow}{X}}, \underset{\text{subcomplex}}{\underset{\downarrow}{A}})$ is a good pair.

Most spaces (smooth manifolds, varieties , etc).
are CW complexes .

Cellular homology.

<u>Lemma</u>: For X a CW complex.

(a) $$H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \mathbb{Z}\{ n\text{-cells of } X \}. & k=n \end{cases}$$

(b) $H_k(X^n)=0 \quad \forall\, k>n$.

(c) $H_k(X^n) \xrightarrow{i_*} H_k(X)$ is an iso $\forall\, k<n$
and surjective for $k=n$.

<u>Say</u>: HW 2.1.22 is very similar to this Lemma.

証 (a) (X^n, X^{n-1}) good pair.

$$\tilde{H}_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n / X^{n-1})$$

$$= X^{n-1} \coprod_\alpha D_\alpha^n \Big/ X^{n-1} = \coprod_\alpha D_\alpha^n \Big/ \underbrace{\coprod_\alpha \partial D_\alpha^n}_{\subseteq X^{n-1}}$$

$$= \bigvee_\alpha S^n.$$

$$X^n \longrightarrow X^n / X^{n-1}$$

(b) LES of (X^n, X^{n-1}):

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \xrightarrow{i_*} H_k(X^n) \to H_k(X^n, X^{n-1})$$

$k \neq n \Rightarrow i_*$ is onto

$k \neq n-1 \Rightarrow i_*$ is 1-1.

So

$$\underbrace{H_k(X^0) \to H_k(X^1) \to \cdots \to H_k(X^{k-1})}_{\text{iso.}} \underbrace{\xrightarrow[\text{1-1}]{} H_k(X^k) \xrightarrow[\text{onto.}]{}} H_k(X^{k+1})$$

$\Downarrow$

(b) since $H_k(X^0)=0$ $\forall k>0$.

If X finite-dim., (c) ✓

If X is ∞-dim., to show surjectivity.

Take $[z] \in H_n(X)$. , z a representative

$z = \sum_\alpha n_\alpha \sigma_\alpha$ $\quad$ σ_α has compact image

so contained in X^N..

then proceed. ...

□.

Let X be a CW complex

$X^0 \subseteq \cdots \subseteq X^n \subseteq \cdots$ gives:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & H_n(X^{n+1}) \arrow[r] & 0 & \\
 & & H_n(X^n) \arrow[ur] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\partial_{n+1}"] \arrow[rrr, "d_{n+1}"] & & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & \\
 & & & 0 \arrow[ur] & &
\end{tikzcd}$$

Define $d_{n+1} = j_n \partial_{n+1}$

exercise: $d_n d_{n+1} = 0$.

Consider $C_n^{cw}(X) := H_n(X^n, X^{n-1}) \quad \forall n$.

$(C_\bullet^{cw}(X), d_\bullet)$ is called the cellular chain complex of X.

note $C_n^{cw}(X) = \mathbb{Z}\{\text{n-cells in } X\}$.

The homology of $C_\bullet^{cw}(X)$ is called the cellular homology group of X, denoted by $H_n^{cw}(X)$.

Thm: For X a CW complex

$$H_n^{CW}(X) = H_n(X)$$

↳ singular homology.

Pf: Look at the diagrams above.

$$H_n(X) \underset{\text{Lemma}}{\cong} H_n(X^{n+1}) = \frac{H_n(X^n)}{\operatorname{Im} \partial_{n+1}}.$$

j_n is injective $\Rightarrow$ j_n maps $\operatorname{Im} \partial_{n+1}$ isomorphically onto $j_n(\operatorname{Im} \partial_{n+1}) = \operatorname{Im} d_{n+1}$.

and maps $H_n(X^n)$ isomorphically onto $\operatorname{Im}(j_n) = \operatorname{Ker}(\partial_n)$.

j_{n-1} is injective $\Rightarrow$ $\operatorname{Ker} \partial_n = \operatorname{Ker} d_n$

Hence,

$$\frac{H_n(X^n)}{\operatorname{Im} \partial_{n+1}} \xrightarrow[j_n]{\cong} \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}} = H_n^{CW}(X).$$

□.

properties of H_n^{cw}

① $H_n(X) = 0$ if X has no n-cells.

② ~~if X has~~ More generally,

rank $H_n(X) \leq$ # of n-cells in X = rank $C_n^{cw}(X)$

③ If X is a cw complex with

no two of its cells in adjacent dimensions,

then $H_n(X) \cong \mathbb{Z}\{n\text{-cells in } X\}$

since $d_n = 0 \ \forall n$.

quick computations.

①,② $\Rightarrow$ ~~$H_n(S^n) = \{ \mathbb{Z}$~~ n $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else}. \end{cases}$

③ $\Rightarrow$ $H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{~~ie~~ } i \text{ even, } i \leq 2n \\ 0 & \text{else}. \end{cases}$

How about $\mathbb{R}P^n$?

We need a formula for d_n.

Formula for d_n:

Thm: For X a CW complex.

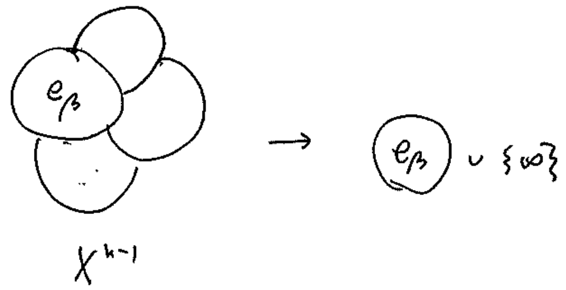
$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the map

$$S^{n-1}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow \frac{X^{n-1}}{X^{n-1}\setminus e^{n-1}_\beta} \cong S^{n-1}_\beta .$$

↳ open disk. $\mathring{D}^{n-1}$

pic:



pf skipped.

intuition: $d_{\alpha\beta}$ = # of times ∂e^n_α wraps around e^{n-1}_β.

(Explain choice of orientation) → $\boxed{\text{STOP}}$

Rmk (Orientation).

An orientation of S^n is a choice of a generator for $H_n(S^n)$.

There is a tiny issue with the d_n formula:

$$S_\alpha^{n-1} \longrightarrow S_\beta^{n-1}$$

Which orientation should we choose for S_α^{n-1}, S_β^{n-1}?
(note: opposite choice will give a sign to degree).

Our choice: pick a generator for

$$\tilde{H}_0(S^0) \cong H_0(D^0, \partial D^0)$$

pick a generator for

$$\begin{tikzcd} H_n(D^n, \partial D^n) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\shortparallel" rotate=90] & H_{n-1}(D^{n-1}, \partial D^{n-1}) \arrow[d, phantom, "\shortparallel"] & \text{inductively.} \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \\ 1 \arrow[r, mapsto] & 1 \end{tikzcd}$$

So we have a consistent ~~choice~~ choice of ~~orientation~~ generators for $H_n(D^n, \partial D^n) \cong \tilde{H}_n(S^n)$ $\forall n$.

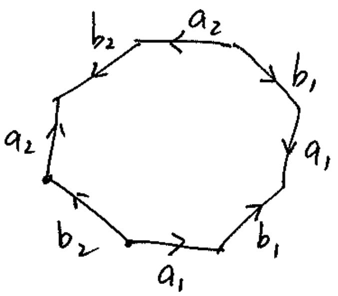
$$\begin{tikzcd} S_\alpha^{n-1} \arrow[r] & S_\beta^{n-1} \\ \text{orientation from } \partial D_\alpha^n \arrow[u] & \text{orientation from } D_\beta^{n-1}/\partial D_\beta^{n-1}. \arrow[u] \end{tikzcd}$$

Computations.

(1) $X = \Sigma_g$

4g-gon with edges identified $\rightsquigarrow$ CW structure on ~~X~~ Σ_g.
with 1 0-cell
2g 1-cells
1 2-cells.

attaching map of 2-cell: $\varphi : S^1 \longrightarrow X^1 = \bigvee_{2g} S^1$
by $[a_1 b_1] \cdots [a_g b_g]$.



$C_\bullet^{CW}$ is:

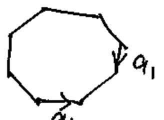
$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \to 0$$

$d_1(e^1_\alpha) = v - v = 0$.

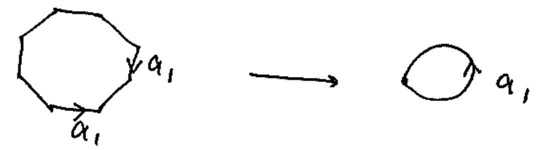
$d_2(e^2) = \sum_\beta d_\beta e^1_\beta$

to compute d_β, take e.g. $e^1_\beta = a_1$.

$$\begin{tikzcd} \partial D^2 = S^1 \arrow[r, "\varphi"] & X^1 \arrow[r] \arrow[d, phantom, "\shortparallel"] & \frac{X^1}{X^1 \setminus a_1} \arrow[d, phantom, "\shortparallel"] \\ & \bigvee_{2g} S^1 & S^1 \end{tikzcd}$$

$S^1 \longrightarrow$  $\longrightarrow$ 

$$\begin{tikzcd}
\partial D^2 \arrow[r, "\varphi"] \arrow[d, phantom, "\shortparallel"] & X^1 \arrow[r] \arrow[d, phantom, "\shortparallel"] & \dfrac{X^1}{X^1 - a_1} \\
S^1 \arrow[r] & {} & {}
\end{tikzcd}$$



$[0,1] \longrightarrow [a_1,b_1]\cdots[a_g,b_g] \longmapsto a_1 a_1^{-1}$ (set everything else to 1)

homotopic to id $\Rightarrow d_\beta = 0$.

So $d_2 = 0$.

$$H_n(\Sigma_g) \cong C_n(\Sigma_g) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}^{2g} & n=1 . \end{cases}$$

Rmk: ~~So~~ This is a "minimal" CW complex structure on Σ_g.

(2) $X = \mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$.

$C^{cw}_\bullet(X)$: $\quad 0 \to \underset{C_n}{\mathbb{Z}} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \to \underset{C_0}{\mathbb{Z}} \to 0.$

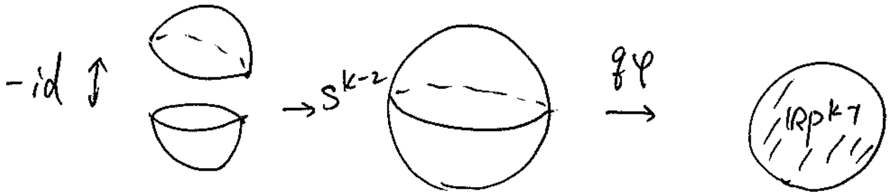
Compute d_k by:

$$S^{k-1} \xrightarrow{\varphi} X^{k-1} = \mathbb{R}P^{k-1} \xrightarrow{q} \cancel{\mathbb{R}P^k}\ \frac{X^{k-1}}{X^{k-1} - e^{k-1}} = \frac{\mathbb{R}P^{k-1}}{\mathbb{R}P^{k-1} - \mathbb{R}^{k-1}}$$

$\varphi: S^{k-1} \longrightarrow \dfrac{S^{k-1}}{\pm 1} = \mathbb{R}P^{k-1}$. $\qquad = \dfrac{\mathbb{R}P^{k-1}}{\mathbb{R}P^{k-2}} \cong S^{k-1}$.

$/\mathbb{R}P^{k-2} \cong$

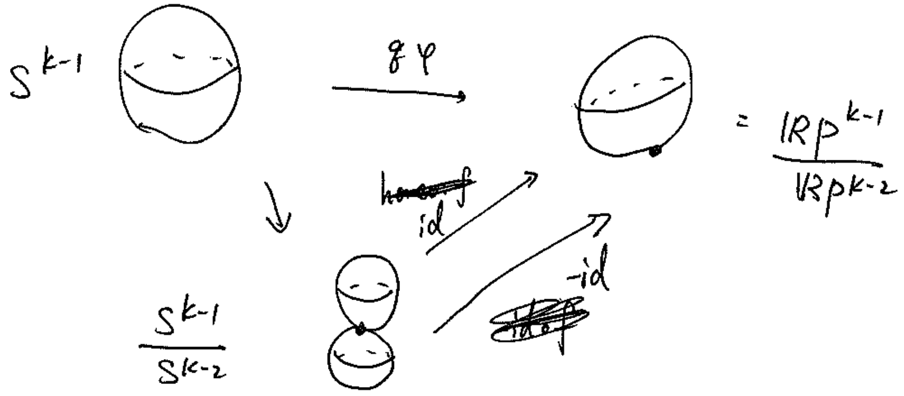
Composition $q\varphi$ restricts to homeomorphisms from each component of $S^{k-1} - S^{k-2} \xrightarrow{q\varphi} \mathbb{R}P^{k-1} - \mathbb{R}P^{k-2}$.



The two homeomorphisms are obtained from each other by precomposing with $-id : S^{k-1} \to S^{k-1}$.

$$\deg(-id) = (-1)^k.$$

Hence, $\deg(q\varphi) = \deg(id) + \deg(-id) = 1 + (-1)^k$.



$$d_k = \begin{cases} 0 & k \text{ odd} \\ 2 & k \text{ even.} \end{cases} \qquad \cdots \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow[d_1]{0} \mathbb{Z} \to 0$$

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k=0 \text{ and } k=n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & k \text{ odd} \quad 0<k<n \\ 0 & \text{else.} \end{cases}$$

Euler characteristic.

X a finite CW complex.

The Euler characteristic of X is

$$\chi(X) := \sum_n (-1)^n \cdot (\#\text{ of } n\text{-cells in } X).$$

Thm: $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$.

$\boxed{\text{Algebra}}$

Lemma: If $(C_\bullet, d)$ is a chain complex,

then $\sum_n (-1)^n \operatorname{rank}(C_n) = \sum_n (-1)^n \operatorname{rank}(H_n(C_\bullet))$.

Pf: key fact: For $0 \to A \to B \to C \to 0$ SES of ab. gp.

$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C)$.

"rank is additive for SES".

Given $(C_\bullet, d)$.

$$H_n = \frac{\operatorname{Ker} d_n}{\operatorname{Im} d_{n+1}} = \frac{Z_n}{B_n}$$

So we have SES's:

$$0 \to Z_n \to C_n \xrightarrow{d_{n+1}} B_{n-1} \to 0$$

$$0 \to B_n \to Z_n \to H_n \to 0$$

$r(C_n) = r(Z_n) + r(B_{n-1})$

$r(Z_n) = r(B_n) + r(H_n)$. $\Rightarrow$ $r(H_n) = r(Z_n) - r(B_n)$.

So $\sum_n (-1)^n r(C_n) = \sum_n (-1)^n r(H_n)$. $\square$.

Rmk: ① The same argument applies to any invariant r of abelian groups that is additive wrt SES's.

② Thm $\Rightarrow$ $\chi(X)$ only depends on the homotopy type of X!

Thm (Poincaré - Hopf).

A smooth compact manifold M has a nonvanishing vector field iff $\chi(M) = 0$.

We have proven the thm for $M = S^n$. $\forall n$.

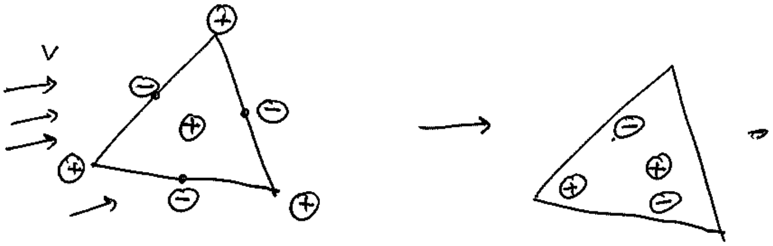
note: $\chi(S^n) = 1 + (-1)^n = 0$ iff n odd.

(W. Thurston)

pf for $M = \Sigma_g$:

Triangulate Σ_g .

Suppose v is a vector field on Σ_g .
(assume $v \pitchfork$ each edge).
put electric charge on each triangle:



Total charge $= \chi(\Sigma_g)$.

Blow the charges along v ("wind") by small ~~along~~ amount.

Total charge $= \sum$ charges in $\overset{\circ}{\Delta} = 0$.

□

Lens spaces.

Def. For $m > 1$, $\ell_1, \cdots, \ell_n$ relatively prime to m,

$$L_m(\ell_1, \cdots, \ell_n) := S^{2n-1} \big/ \mathbb{Z}_m$$

where $\mathbb{Z}_m \curvearrowright S^{2n-1} \subseteq \mathbb{C}^n$ via $\rho : S^{2n-1} \to S^{2n-1}$.

$$\rho(z_1, \cdots, z_n) = (\zeta^{\ell_1} z_1, \zeta^{\ell_2} z_2, \cdots, \zeta^{\ell_n} z_n),$$

"lens space". $\zeta := e^{\frac{2\pi i}{m}}$, primitive m-th root of 1.

Rmk: ① When $m=2$, ρ is the antipodal map.

So $L_2(1, \cdots, 1) = \mathbb{R}P^{2n-1}$.

② In general, ρ has no fixed point

$$\Rightarrow \quad S^{2n-1} \xrightarrow{q} S^{2n-1} / \mathbb{Z}_m \text{ is a covering map } (m:1).$$

If $n > 1$, S^{2n-1} is simply connected.

so q is a universal cover.

$$\Rightarrow \quad \pi_1(L_m) \cong \mathbb{Z}_m.$$

L_m is a compact $(2n-1)$-dim. manifold.

(e.g. $L_7(1,1)$ and $L_7(1,2)$).

③ $\exists$ two lens spaces that are homotopy equivalent but not homeomorphic!

(Poincáre conjecture: This cannot ~~be~~ happen for spheres).

④ $L_m(l_1, \cdots, l_n)$ has a CW structure with one k-cell for all $k \le 2n-1$.

cellular chain complex:

$$0 \to \underset{\underset{C_{2n-1}}{\shortparallel}}{\mathbb{Z}} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \underset{\underset{C_0}{\shortparallel}}{\mathbb{Z}} \to 0.$$

Hence
$$H_k(L_m(l_1, \cdots, l_n)) = \begin{cases} \mathbb{Z} & k=0,\ 2n-1 \\ \mathbb{Z}_m & k \text{ odd},\ 0<k<2n-1 \\ 0 & \text{else}. \end{cases}$$

This generalizes the case for $\mathbb{R}P^{2n-1}$.
(Read Hatcher p.145 for details).

Note: $H_k(L_m)$ is independent of l_i's. (also π_1).

⑤ We can also have ∞-dimensional lens space

$$L_m(l_1, l_2, \dots) = S^\infty / \mathbb{Z}_m$$

an ∞-dimensional CW complex.

Then $H_k(L_m) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/m & k \text{ odd} \\ 0 & \text{else}. \end{cases}$

<u>Def</u>: ~~A spa~~ For G a group,

a path-connected space with $\pi_1 \cong G$ and with a contractible universal cover is called a <u>$K(G,1)$ space</u>. or an Eilenberg-Maclane space for G.

Sewy $1 \rightsquigarrow \pi_1$.

<u>Examples</u>:

- $L_m = S^\infty/\mathbb{Z}_m$ is a $K(\mathbb{Z}_m, 1)$ space.
  in particular, $\mathbb{R}P^\infty = S^\infty/\mathbb{Z}_2$ is a $K(\mathbb{Z}_2, 1)$ space.
- S^1 is a $K(\mathbb{Z},1)$ space
- $\bigvee_k S^1$ is a $K(F_k, 1)$ space
  $\hookrightarrow$ free group.
- Σ_g is a $K(G,1)$ space for $G = \pi_1(\Sigma_g)$. $(g>0)$.

Same holds for any compact, connected, nonpositively curved manifold M. (Cartan-Hadamard).

Fact: For any G, $\exists$ a CW complex that is a $K(G,1)$ space.

Moreover, such CW complex is unique up to homotopy.

Homology of groups.

$$H_n(G) = H_n(K(G,1)).$$

$\hookrightarrow$ determined by the group G.

Prop: If X is a finite-dimensional CW complex and a $K(G,1)$, then $G = \pi_1 X$ is torsion free.

Pf: Suppose not. Then $\mathbb{Z}_m \subseteq G$ for some $m > 1$.

Then $\tilde{X}/\mathbb{Z}_m$ is a $K(\mathbb{Z}_m, 1)$ space.

$$\begin{array}{c} \tilde{X}/\mathbb{Z}_m \\ \downarrow \\ X = \tilde{X}/G \quad \text{fin-dim}. \end{array}$$

$\Rightarrow$ $\tilde{X}/\mathbb{Z}_m$ is also a finite-dim. CW complex.

HR $\Rightarrow$ $H_k(\tilde{X}/\mathbb{Z}_m) = 0$ for k large enough.

However, $H_k(\tilde{X}/\mathbb{Z}_m) \cong H_k^{\text{group}}(\mathbb{Z}_m) = H_k(L_m) \neq 0 \quad \forall k$ odd. $\square$

Cor: $\pi_1(\Sigma_g)$ is torsion free.

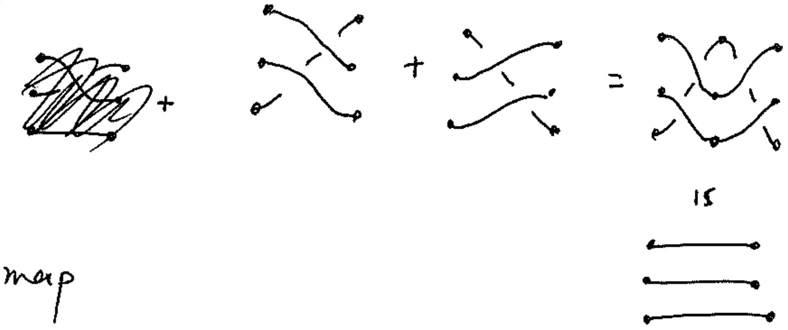
Same for $\pi_1(M)$, M non positively curved.

Applications to braid groups

The braid group on n strands. B_n consists of braids (up to isotopy).

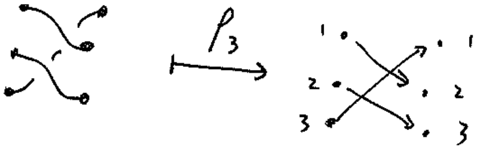
e.g.  $\in B_3$.

group operation:



There is a group map

$\rho_n : B_n \twoheadrightarrow S_n$.

e.g. 

$= (1 2 3) \in S_3$.

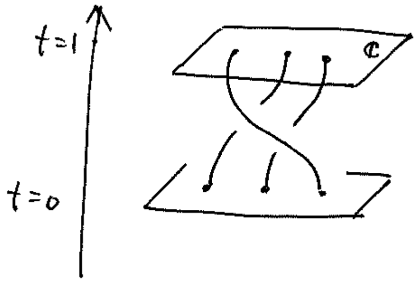
The pure braid group is $P_n = \ker \rho_n \leq B_n$.

The <u>ordered configuration space</u> of n points in $\mathbb{C}$.

is $$\mathrm{PConf}^n\mathbb{C} = \left\{ (z_1, \cdots, z_n) \;\middle|\; \begin{matrix} z_i \in \mathbb{C} & \forall i \\ z_i \neq z_j & \forall i \neq j \end{matrix} \right\}.$$
$$= \mathbb{C}^n \setminus \bigcup_{i \neq j} \{ z_i = z_j \}$$ a manifold.
~~finite~~ 2n-dimensional.

<u>observe</u>: $\pi_1(\mathrm{PConf}^n\mathbb{C}) \cong P_n$.

a loop in $\mathrm{PConf}^3\mathbb{C}$: $\rightsquigarrow$ a 3-braid.



~~Def~~ The <u>unordered configuration space</u> of n pts in $\mathbb{C}$.

is $\mathrm{UConf}^n\mathbb{C} = \mathrm{PConf}^n\mathbb{C} / S_n$.

<u>note</u>: $S_n \circlearrowright \mathrm{PConf}^n\mathbb{C}$ freely.

$$\begin{tikzcd} \mathrm{PConf}^n\mathbb{C} \arrow[d, "S_n"] \\ \mathrm{UConf}^n\mathbb{C} \end{tikzcd}$$ a normal cover.
with deck group S_n.

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(\mathrm{PConf}^n\mathbb{C}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \pi_1(\mathrm{UConf}^n\mathbb{C}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & S_n \arrow[r] & 0 \\ 0 \arrow[r] & P_n \arrow[r] & B_n \arrow[r, "f_n"] & S_n \arrow[r] & 0. \end{tikzcd}$$

Fact: $P\mathrm{Conf}^n\mathbb{C}$ has contractible universal cover.

So $P\mathrm{Conf}^n\mathbb{C}$ is a $K(P_n, 1)$

$U\mathrm{Conf}^n\mathbb{C}$ is a $K(B_n, 1)$.

Prop: P_n and B_n are torsion-free.

Pf: $P\mathrm{Conf}^n\mathbb{C}$ is a fin. dim. smooth manifold

$\overset{\text{Morse theory}}{\Longrightarrow}$ $P\mathrm{Conf}^n\mathbb{C}$ is homotopy equivalent to a fin. dim. CW complex.

$\overset{\text{prop.}}{\Longrightarrow}$ $\pi_1(P\mathrm{Conf}^n\mathbb{C})$ is torsion free. $\square$.

Cor:

$$\begin{tikzcd} 0 \arrow[r] & P_n \arrow[r] & B_n \arrow[r, "p_n"] & S_n \arrow[r] \arrow[l, bend left, dashed, "\nexists\, s"] & 0 \end{tikzcd}$$

does not have a section $s : S_n \to B_n$ s.t. $p_n \circ s = id$.

Mayer - Vietoris sequences.

Suppose $X = A \cup B$.

Q: What is $H_n(X)$ in terms of $H_n(A)$, $H_n(B)$?

Guess: $H_n(X) \overset{?}{=} \dfrac{H_n(A) \oplus H_n(B)}{H_n(A \cap B)}$.

Not true, but close.

Thm. Given $A, B \subseteq X$ s.t. $\mathring{A} \cup \mathring{B} = X$, then $\exists$ a LES ("Mayer-Vietoris sequence")

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

Pf: Recall in our proof of excision, we define

$$C_n(A+B) = \mathbb{Z}\{\sigma : \Delta^n \to X \mid \operatorname{im}(\sigma) \subseteq A \text{ or } \operatorname{im}(\sigma) \subseteq B\}$$
$$= C_n(A) + C_n(B) \subseteq C_n(X).$$

claim: $\exists$ a SES of chain complexes

$$0 \to C_n(A \cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0.$$

where $\varphi(x) = (x, -x)$. $\psi(x, y) = x + y$.

Pf is quick.

SES of chain complex ⇝ LES of homology.

<u>note</u>: $H_\bullet(C_\bullet(A) \oplus C_\bullet(B)) = H_\bullet(C_\bullet(A)) \oplus H_\bullet(C_\bullet(B))$.
(explain).

Before, we had: $C_n(A+B) \hookrightarrow C_n(X)$ induces iso. on homology. (provided $\mathring{A} \cup \mathring{B} = X$).

$\square$.

<u>Q</u>: what are the maps in the LES ?

The first two are induced by φ, ψ.

$\partial : H_n(X) \longrightarrow H_{n-1}(A \cap B)$.

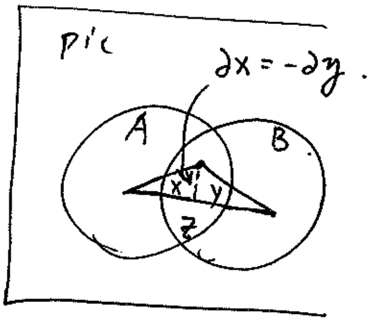
Take $z \in C_n(X)$ representing $[z] \in H_n(X)$.

$\partial z = 0$ in $C_{n-1}(X)$.

choose x, y s.t. $z = x+y$, $x \in C_n(A)$, $y \in C_n(B)$.

$0 = \partial z = \partial(x+y) = \partial x + \partial y$.

$\partial[z]$ is $[\partial x] = [-\partial y]$.



Rmk: Thm holds if we replace H by $\tilde{H}$

(your job)

Example: If $A, B, X, A\cap B$ are path connected (so $\tilde{H}_0 = 0$).

then

$$H_1(A\cap B) \xrightarrow{\Phi} H_1(A)\oplus H_1(B) \to H_1(X) \to 0$$

so $$H_1(X) = \frac{H_1(A)\oplus H_1(B)}{\Phi(H_1(A\cap B))}$$

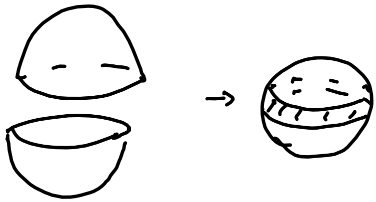
"The abelinization of van Kampen thm"

Recall HW: $H_1 = \pi_1^{ab}$.

Example: $X = S^n$

A, B = hemispheres

$A \cap B$ = a neighborhood of $S^{n-1} \simeq S^{n-1}$.



$\tilde{H}_n(A) \oplus \tilde{H}_n(B) = 0 \quad \forall n$

$\Rightarrow$

$$\begin{tikzcd}
\tilde{H}_n(X) \arrow[r, "\partial"] \arrow[d, equal] & \tilde{H}_{n-1}(A \cap B) \arrow[d, phantom, "\rotatebox{90}{$\cong$}"] \\
\tilde{H}_n(S^n) & \tilde{H}_{n-1}(S^{n-1})
\end{tikzcd}$$

is iso.

$\Rightarrow \tilde{H}_n(S^n) = \mathbb{Z} \quad \forall n$. by induction.

Homology with coefficients.

We can replace $\mathbb{Z}$ by an abelian group G. in the definition of H_n.

$$C_n(X,G) := G\{\sigma : \Delta^n \to X\}.$$

$$H_n(X;G) := H_n(C_\cdot(X,G), \partial)$$

"homology of X with coefficients in G."

The theory for $H_n(X;G)$ is the same as before.

Computations might be different.

Ex: $G = k$ (a field), $X = \mathbb{R}P^2$.

$$C_\cdot(\mathbb{R}P^2, k): \quad \cdots \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k \to 0$$

if char $k=2$. $2=0$ in k.

$$H_i(\mathbb{R}P^2; k) = 0 \quad \forall\, i \le n.$$

if char $k = p \neq 2$. ~~2~~ $\frac{1}{2} \in k$.

$$H_i(\mathbb{R}P^2; k) = \begin{cases} k & i=0,1 \\ 0 & \text{else} \end{cases}.$$

HW:

$$\chi(X) = \sum_i (-1)^i \dim_k H^i(X;k)$$

Category theory basics.

Def. A category $\mathcal{C}$ consists of

(1) a collection $ob(\mathcal{C})$ of objects

(2) $\forall X, Y \in ob(\mathcal{C})$, a set $Mor(X, Y)$ of maps, or morphisms with a distinguished $id_X \in Mor(X, X)$, $\forall X$.

(3) A composition function

$$Mor(X, Y) \times Mor(Y, Z) \longrightarrow Mor(X, Z)$$
$$(f, g) \longmapsto g \circ f$$

s.t. $f \circ id = f$, $id \circ f = f$, $(f \circ g) \circ h = f \circ (g \circ h)$.

Examples:

- Category of top. spaces (or CW complexes) with continuous maps.
- Cat. of groups. (or abelian groups) with homomorphisms.
- Cat. of sets
- Cat. of vector spaces (or modules over R).
- Every group G can be viewed as a category $\mathcal{G}$ with one object ob and $G = \mathrm{Mor}(ob, ob)$.

<u>Def</u>: A (co<u>variant) functor</u> F from $\mathcal{C}$ to $\mathcal{D}$

assigns $X \in ob(\mathcal{C}) \longmapsto F(X) \in ob(\mathcal{D})$

and $f \in Mor_{\mathcal{C}}(X, Y) \longmapsto F(f) \in Mor_{\mathcal{D}}(FX, FY)$

s.t. $F(f \circ g) = F(f) \circ F(g)$

$F(\underset{\mathcal{C}}{id_X}) = id_{F(X)} \quad \forall X \in ob(\mathcal{C})$

A <u>contravariant functor</u> F from $\mathcal{C}$ to $\mathcal{D}$

assigns $X \longmapsto F(X)$

$f \in Mor_{\mathcal{C}}(X, Y) \longmapsto F(f) \in Mor_{\mathcal{D}}(FY, FX)$

or equivalent, a functor from $\mathcal{C}^{op}$ to $\mathcal{D}$ ..

Example: For each n,

$$\begin{tikzcd}
\{\text{top. spaces}\} \arrow[r] & \{\text{ab. groups}\} \\
X \arrow[r, mapsto] & H_n(X)
\end{tikzcd}$$

is a functor.

In fact, $H_n(-)$ is the composition of two functors:

$$\begin{tikzcd}
\{\text{top. spaces}\} \arrow[r, "\boxed{\text{topology}}"] & \left\{\begin{array}{c}\text{chain complexes}\\ \text{and chain maps}\end{array}\right\} \arrow[r, "\boxed{\text{algebra}}"] & \{\text{ab. groups}\} \\
X \arrow[r] & (C_\bullet(X), \partial) \arrow[r] & H_n(C_\bullet(X), \partial)
\end{tikzcd}$$

or

$$\begin{tikzcd}
\left\{\begin{array}{c}\Delta\text{-complexes}\\ \text{and } \Delta\text{-maps.}\end{array}\right\} \arrow[r] & \{\text{c.c}\} \arrow[r] & \{\text{ab. gp.}\}. \\
X \arrow[r, mapsto] & (C_\bullet^\Delta(X), \partial) \arrow[r] & H_n(C_\bullet^\Delta(X), \partial) \arrow[d, phantom, "\shortparallel"] \\
 & & H_n^\Delta(X).
\end{tikzcd}$$

Say: Similarly for pairs (X, A).

Example 2

$$\begin{aligned} \{\text{vect. spaces}\} &\longrightarrow \{\text{Vector. spaces}\} \\ V &\longmapsto V^* = \mathrm{Hom}(V, k) \end{aligned}$$

is a contravariant functor.

Example 3:

Suppose a group G acts on a set X. Then we have a functor

$$\begin{aligned} G &\longrightarrow Set \\ ob &\longmapsto X. \end{aligned}$$

$$\begin{tikzcd} ob \arrow[loop below, "g"] \arrow[r, mapsto] & X \arrow[loop below, "g"] \end{tikzcd}$$

Say: a representation.

Def: Suppose $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ are two functors, a natural transformation T from F to G

assigns

$$X \in \mathrm{ob}(\mathcal{C}) \longmapsto T_X \in \mathrm{Mor}_{\mathcal{D}}(F(X), G(X))$$

s.t. $\forall X, Y$,

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Example 1:

$$\{\text{pairs } (X,A)\} \longrightarrow \{\text{ab. groups}\}.$$

$$(X,A) \longmapsto H_n(X,A)$$

$$(X,A) \longmapsto H_{n-1}(A).$$

The boundary map

$$H_n(X,A) \xrightarrow{\partial} H_{n-1}(A)$$

is a nat. trans.

Example 2:

$$\{\text{vect. spaces}\} \longrightarrow \{\text{vect. spaces}\}$$

$$V \longmapsto V$$

$$V \longmapsto V^{**} = \mathrm{Hom}(\mathrm{Hom}(V,k),k)$$

$$ev: V \longrightarrow V^{**}$$

$$v \longmapsto ev(v): V^* \longrightarrow k$$

$$\varphi \longmapsto \varphi(v)$$

is a nat. trans.

Eilenberg-Steenrod axioms:

A homology theory consists of a sequence of functors H_n, $n \in \mathbb{Z}_{\geq 0}$

$$\left\{\begin{matrix}\text{pairs of top.} \\ \text{spaces } (X,A)\end{matrix}\right\} \longrightarrow \{\text{ab. gp.}\}.$$

$$(X,A) \longmapsto H_n(X,A)$$

and a natural transformation

$$\partial : H_n(X,A) \to H_{n-1}(A) := H_{n-1}(A,\phi)$$

satisfying the following axioms:

1. (Homotopy invariance):

   if $f, g : (X,A) \to (Y,B)$ are homotopic then $f_* = g_*$

2. (Excision).

3. (Additivity).

$$H_n(\coprod_\alpha X_\alpha) = \bigoplus_\alpha H_n(X_\alpha)$$

4. (LES of pairs)

5. (Dimension):

$$H_n(pt) = 0 \quad \forall n \neq 0.$$

Thm: Any two homology theory satisfying those axioms are isomorphic.

why plausible?

Those axioms $\Rightarrow$ $H_\bullet(S^n)$ $\forall n$.

$\Rightarrow$ theory of degree

$\Rightarrow$ $H_\bullet$ of CW complexes.

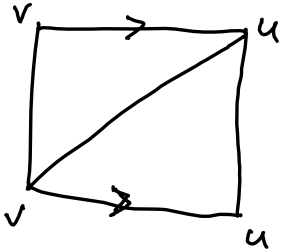
$\Rightarrow$ $H_\bullet$ of top. spaces.

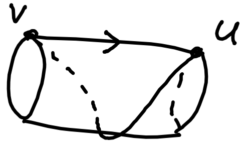
Simplicial approximation (Hatcher 2C).

A simplicial complex is a Δ-complex whose simplices are uniquely determined by their vertices.

Fact: Every Δ-complex becomes a simplicial after some subdivision.

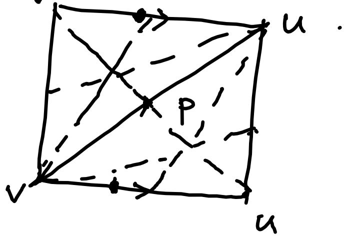
e.g.





not a s.s. b/c $[u,v]$ represents 2 different simplies.

↓ subdivide



Suppose K, L are simplicial complexes

a map $f: K \to L$ is simplicial if it maps each simplex of K to a simplex of L by a linear map taking vertices to vertices.

(so f is uniquely determined by its value on vertices).

Thm. (Simplicial approximation thm).

If K is a finite simplicial complex, then any map $f: K \to L$ is homotopic to a map that is simplicial w.r.t. some iterated barycentric subdivison of K.

pf skipped.

Lefschetz Fixed Point Thm.

X = a space.

Q: when does $f: X \to X$ have a fixed pt ?

Consider $f_* : \underbrace{H_n(X)}_{\text{torsion}} \to \underbrace{H_n(X)}_{\text{torsion}}$.

f_* is a matrix with $\mathbb{Z}$-entries.

$\mathrm{Tr}(f_*) = \sum$ diagonal entries of f_*

Def: The Lefschetz number of f is

$$\Lambda(f) := \sum_n (-1)^n \, \mathrm{Tr}\left(f_* : \frac{H_n(X)}{\text{tor}} \to \frac{H_n(X)}{\text{tor}} \right).$$

Thm. (Lefshetz Fixed Pt thm).

For X a finite simplicial complex, if $\Lambda(f) \neq 0$, then f has a fixed pt.

Rmk1: Lefshetz for $X = D^n \Rightarrow$ Brouwer F.T.

Rmk2: Compactness is necessary!

Take $X = \mathbb{R}$. $f: \mathbb{R} \longrightarrow \mathbb{R}$ translate by 1.

$\Lambda(f) = 1$ ($f_* = id$ on H_0).

But f has no fixed pt!

Note: f fixes ∞ on $\mathbb{R} \cup \{\infty\} = S^1$..

Rmk 3. Thm is true for X = a finite CW complex or compact manifolds.

X finite CW $\Rightarrow$ X compact, locally contractible can be embedded in $\mathbb{R}^N$.

$\Rightarrow$ X = a retract of a finite simplicial complex.

$$\begin{tikzcd} X \arrow[r, hook, "i"'] & K \arrow[l, dashed, bend right, "r"'] \end{tikzcd}$$

$f \curvearrowright X \rightsquigarrow fr \curvearrowright K$

$\Lambda(f) = \Lambda(fr)$

fr has a fix. pt. $\Rightarrow$ f has a fix pt.

Pf: Suppose $f(x) \neq x \quad \forall x \in X$.

Put a metric on X.

Since X is compact, $\exists\, \delta > 0$ s.t.

$$d(f(x), x) > \delta \quad \forall x \in X.$$

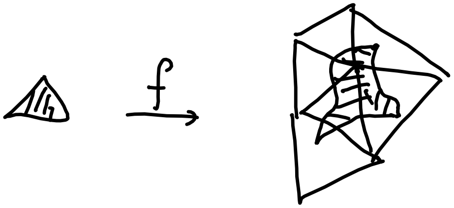
We barycentric subdivide X s.t. each simplex has diameter $< \frac{\delta}{100}$.

By simplicial approximation, after barycentric subdivisions, $\exists\, h : X \to X$ simplicial map

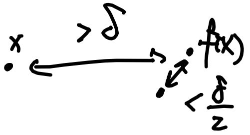
s.t. ① $h \simeq f$

② $d(h(x), f(x)) \leq \frac{\delta}{2} \quad \forall x \in X$.

pic:



$\forall x \in X,$



$$d(h(x),x) > \frac{\delta}{2} > \frac{\delta}{100}$$

h simplicial $\Rightarrow$ $h(\sigma)\cap\sigma = \phi$ fr all simplices σ.

<u>Say</u>: (idea) f moves every point

$\Rightarrow$ h moves every simplices.

Consider

$$\begin{array}{ccc} h_\# : C_n^\Delta(X) & \longrightarrow & C_n^\Delta(X). \\ \parallel & & \\ \mathbb{Z}\{n\text{-simplices in } X\}. & & \\ \circlearrowleft & & \\ h. & & \end{array}$$

$$\mathrm{Tr}\left(h_\# : C_n^\Delta(X) \to C_n^\Delta(X)\right) = \left(\begin{array}{l}\#\text{ of } n\text{-simplices } \sigma \\ \quad \text{s.t. } h(\sigma)=\sigma\end{array}\right)$$

$$= 0.$$

Thm follows from the folowing lemma from algebra:

Lemma (Hopf trace formula).

Suppose $(C_\bullet, \partial)$ is a finite-length chain complex s.t. C_n = finite rank free ab. group $\forall n$.
and $f_\#: C_\bullet \to C_\bullet$ is a chain map.
Then

$$\sum_n (-1)^n \operatorname{Tr}(f_\# : C_n \to C_n)$$

$$= \sum_n (-1)^n \operatorname{Tr}(f_* : H_n^{\text{free}}(C_\bullet) \longrightarrow H_n^{\text{free}}(C_\bullet))$$

Pf: Trace is additive w.r.t. SES of ab. gps:

i.e.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 & \text{SES.} \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 & \text{SES} \end{tikzcd}$$

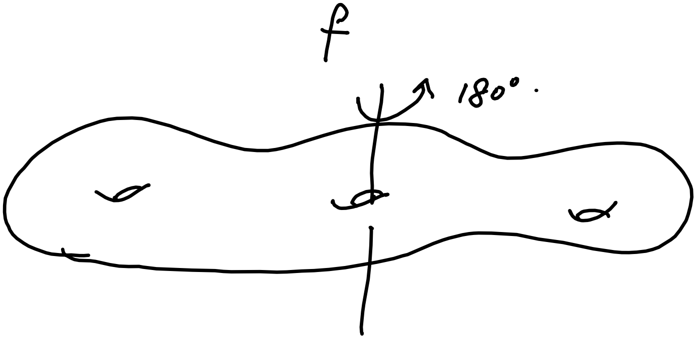
$$\operatorname{Tr}(\beta) = \operatorname{Tr}(\alpha) + \operatorname{Tr}(\gamma) \qquad (\operatorname{Tr} = \text{trace on free parts})$$

Apply our proof that

$$\chi = \sum_n (-1)^n \operatorname{rank} H_n.$$

$\boxed{\text{STOP}}$ $\square$

Example:

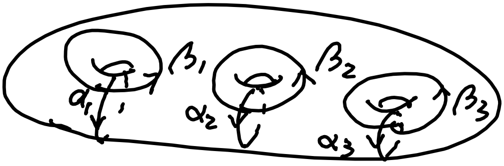
$\Sigma_3 =$ 

f has no fixed pt $\Rightarrow \Lambda(f) = 0$.

$f_* = id$ on $H_0 \cong \mathbb{Z}$

$H_2 \cong \mathbb{Z}$.

$H_1 = \mathbb{Z}\{[\alpha_i], [\beta_i] \mid i = 1, 2, 3\}$.

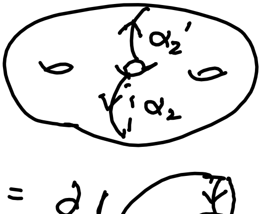


f_* : $[\alpha_1] \longleftrightarrow [\alpha_3]$

$[\beta_1] \longleftrightarrow [\beta_3]$

$[\beta_2] \longmapsto [\beta_2]$

$[\alpha_2] \longmapsto [\alpha_2'] \sim [\alpha_2]$



since $\alpha_2 - \alpha_2' = \partial($  $)$

$\mathrm{Tr}(f_* : H_1 \to H_1) = 2$.

$$\Lambda(f) = \underset{H_0}{\underset{\uparrow}{1}} - \underset{H_1}{\underset{\uparrow}{2}} + \underset{H_2}{\underset{\uparrow}{1}} = 0.$$

Thm (Chevalley-Weil)

Suppose $\begin{matrix} \tilde{\Sigma} \\ \downarrow \\ \Sigma_g \end{matrix}$ is a finite normal cover with deck group G.

Then $H_1(\tilde{\Sigma};\mathbb{Q}) \cong \mathbb{Q}[G]^{2g-2} \oplus \mathbb{Q}^2_{triv}$

as $\mathbb{Q}[G]$-modules.

Rmk $G \curvearrowright \tilde{\Sigma} \Rightarrow G \curvearrowright H_n(\tilde{\Sigma};\mathbb{Q}) \ \forall n.$

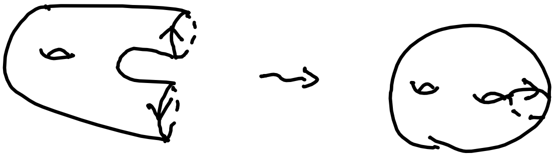
Easy: $\left.\begin{matrix} G \curvearrowright H_0(\tilde{\Sigma};\mathbb{Q}) = \mathbb{Q} \\ G \curvearrowright H_2(\tilde{\Sigma};\mathbb{Q}) = \mathbb{Q} \end{matrix}\right\}$ trivially.

Back to example:

$G = \mathbb{Z}/2\mathbb{Z} = \langle f \rangle$

$\tilde{\Sigma} = \Sigma_3$

$\Sigma_g = ?$



$g=2.$

we had $H_1(\tilde{\Sigma};\mathbb{Q}) = \mathbb{Q}^6$

$\langle \alpha_1, \alpha_3 \rangle \cong \mathbb{Q}[G]$

f fixes $\frac{1}{2}(\alpha_1+\alpha_3)$ and invent $\frac{1}{2}(\alpha_1-\alpha_3)$.

similarly, $\langle \beta_1, \beta_3 \rangle \cong \mathbb{Q}[G]$

$\langle \beta_2 \rangle \cong \mathbb{Q} \circlearrowleft f$ trivially

$\langle \alpha_2 \rangle \cong \mathbb{Q}$

$H_1(\tilde{\Sigma};\mathbb{Q}) \cong \mathbb{Q}[G]^2 \oplus \mathbb{Q}^2$

pf of Thm: Make Σ_g a simplicial complex

with V many vertices

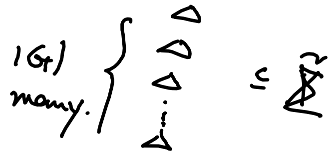
E ... edges

F ... 2-simplices.

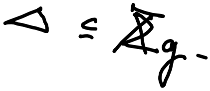
$$V - E + F = \chi(\Sigma_g) = 2-2g.$$

Subdivide Σ_g s.t. each simplex is evenly covered by $\tilde{\Sigma} \to \Sigma_g$

This makes $\tilde{\Sigma}$ into a finite simplicial complex.



Given a simplex σ of Σ_g



$G \curvearrowright \{$ simplex $\tilde{\sigma}$ of $\tilde{\Sigma}$ lifting $\sigma \}$ freely and transitively.

Hence,

$$\left.\begin{aligned} C_0^{\Delta}(\tilde{\Sigma}) &\cong \mathbb{Q}[G]^V \\ C_1^{\Delta} &\cong \mathbb{Q}[G]^E \\ C_2^{\Delta} &\cong \mathbb{Q}[G]^F \end{aligned}\right\} \text{ as } \mathbb{Q}[G]\text{-modules.}$$

$$\begin{tikzcd}
0 \arrow[r] & C_2^{\Delta}(\tilde{\Sigma};\mathbb{Q}) \arrow[r, "\partial_2"] \arrow[d, equal] & C_1^{\Delta}(\tilde{\Sigma};\mathbb{Q}) \arrow[r, "\partial_1"] \arrow[d, equal] & C_0^{\Delta}(\tilde{\Sigma};\mathbb{Q}) \arrow[r] \arrow[d, equal] & 0 \\
& \mathbb{Q}[G]^F & \mathbb{Q}[G]^E & \mathbb{Q}[G]^V &
\end{tikzcd}$$

"maps are G-equivariant."

We know:

① $H_0^{\Delta} = \dfrac{C_0^{\Delta}}{\operatorname{Im}\partial_1}$

so $0 \to \operatorname{Im}\partial_1 \to C_0^{\Delta} \to H_0^{\Delta} \to 0$

SES of $\mathbb{Q}[G]$-modules.

Fact☆: Every SES of $\mathbb{Q}[G]$-module splits if G is finite.

$\Rightarrow C_0^{\Delta} = \operatorname{Im}\partial_1 + H_0^{\Delta}$ (written additively).

$\Rightarrow \operatorname{Im}\partial_1 = C_0^{\Delta} - H_0^{\Delta}$

$0 \to \ker\partial_1 \to C_1^{\Delta} \longrightarrow \operatorname{Im}\partial_1 \to 0$

$$\Rightarrow \ker\partial_1 = C_1^{\Delta} - \operatorname{Im}\partial_1$$

$$= C_1^{\Delta} - C_0^{\Delta} + H_0^{\Delta}$$

② $H_2^\Delta = \ker \partial_2$.

$$\begin{tikzcd}
 & H_2^\Delta \arrow[d, phantom, "\parallel"] & & & \\
0 \arrow[r] & \ker\partial_2 \arrow[r] & C_2^\Delta \arrow[r, "\partial_2"] & \operatorname{Im}\partial_2 \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow \operatorname{Im}\partial_2 = C_2^\Delta - H_2^\Delta$.

① + ② $\Rightarrow$

$$H_1^\Delta = \ker\partial_1 - \operatorname{Im}\partial_2$$

$$= \left(C_1^\Delta - C_0^\Delta + H_0^\Delta\right) - \left(C_2^\Delta - H_2^\Delta\right)$$

$$= \left(-C_0^\Delta + C_1^\Delta - C_2^\Delta\right) + H_0^\Delta + H_2^\Delta$$

$$\cong \mathbb{Q}[G]^{-V+E-F} + \mathbb{Q} + \mathbb{Q}$$

$$\cong \mathbb{Q}[G]^{2g-2} + \mathbb{Q}^2.$$

□.

Exercise: (For those who have taken a course on rep theory):

Give another proof of Thm by computing the character of $G \curvearrowright H_1(\tilde{\Sigma}; \mathbb{Q})$.'

Exercise. (For those who have taken a course on complex analysis):

Generalize the theorem to

$$\begin{tikzcd}\tilde{\Sigma} \arrow[d] \\ \Sigma_g\end{tikzcd}$$

a branch cover.

(use Riemann - Hurwitz).

Cohomology.

X = a topological space.

G = an ab. group.

singular chain complex $(C_\bullet(X), \partial)$:

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

apply the contravariant functor $\mathrm{Hom}(-, G)$

[note: $\mathrm{Ab\,Gp} \longrightarrow \mathrm{Ab\,Gp}$

$$A \longmapsto \mathrm{Hom}(A,G) = \{\varphi : A \to G \mid \text{homo}\}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] & & A \arrow[d, "f"'] \arrow[dr, dashed, "\varphi f \in \mathrm{Hom}(A,G)."] & \\
B \arrow[rr, phantom, "\longmapsto"] & & B \arrow[r, "\varphi \in \mathrm{Hom}(B.G)"'] & G
\end{tikzcd}$$

].

Define $C^n(X;G) := \mathrm{Hom}(C_n(X), G)$

$$\cdots \leftarrow C^{n+1}(X;G) \xleftarrow{\delta_{n+1}} C^n(X;G) \xleftarrow{\delta_n} C^{n-1}(X;G) \leftarrow \cdots$$

Note: δ^2 = dual of ∂^2 $= 0$.

$(C^n(X;G), \delta)$ is called the singular cochain complex of X with coefficients in G.

The singular cohomology of X with ---- in G

is $$H^n(X;G) := \frac{\ker \delta_{n+1}}{\operatorname{Im} \delta_n}$$

$\ker \delta_{n+1} = \{$ cocycles $\}$

$\operatorname{Im} \delta_n = \{$ coboundaries $\}$

Formula for $\delta : C^n \longrightarrow C^{n+1}$.

Given $\varphi \in C^n = \mathrm{Hom}(C_n ; G)$.

$$\delta\varphi = \left(\begin{tikzcd} C_{n+1} \arrow[r, "\partial"] \arrow[rr, bend right] & C_n \arrow[r, "\varphi"] & G \end{tikzcd} \right)$$

$\forall$ $(n+1)$-simplex $\sigma : \Delta^{n+1} \longrightarrow X$, $\sigma \in C_{n+1}$.

$$\delta\varphi(\sigma) = \varphi(\partial\sigma)$$

$$= \varphi\left(\sum_i (-1)^i \, \sigma|_{[v_0 \cdots \hat{v_i} \cdots v_{n+1}]} \right)$$

$$= \sum_i (-1)^i \, \varphi(\sigma|_{[v_0 \cdots \hat{v_i} \cdots v_{n+1}]}).$$

prop: A map $X \xrightarrow{f} Y$ induces a map $H^n(X;G) \xleftarrow{f^*} H^n(Y;G)$ $\forall n$.

$H^n(-;G)$ is a contravariant functor

$$\begin{tikzcd}
\{\text{top. sp.}\} \arrow[rr] \arrow[dd] & & \{\text{ab. gp.}\} \\
X \arrow[r, mapsto] & H^n(X;G) & \\
\{\text{chain complexes}\} \arrow[rr, "\mathrm{Hom}(-,G)"] & & \{\text{cochain complexes}\} \arrow[uu, "H^n"'] \\
C_\bullet(X) \arrow[rr, mapsto] & & C^\bullet(X;G) := \mathrm{Hom}(C_\bullet(X);G)
\end{tikzcd}$$

properties of H_n extend to H^n:

1. reduced cohomology.

$$\xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\rightsquigarrow \ Hom(-,G)$$

$$\xleftarrow{\delta_1} C^0(X;G) \xleftarrow{\varepsilon} Hom(\mathbb{Z},G) \cong G \leftarrow 0 .$$

$$\tilde{H}^0(X;G) := \frac{\ker \delta_1}{\operatorname{Im} \varepsilon}$$

2. relative cohomology: $A \subseteq X$.

$$0 \to C_n(A) \to C_n(X) \longrightarrow C_n(X,A) \to 0 \qquad \text{SES}$$

$$\rightsquigarrow \ \text{apply } Hom(-,G)$$

$$0 \leftarrow C_n(A;G) \leftarrow C_n(X;G) \leftarrow C_n(X,A;G) \leftarrow 0 .$$

SES*.

Rmk$^{☆}$: In general, the functor $\mathrm{Hom}(-,G)$ is only left-exact ("maps kernel to cokernel").

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \quad \text{exact}$$

$$\Rightarrow \quad 0 \to \mathrm{Hom}(C,G) \xrightarrow{j^*} \mathrm{Hom}(B,G) \xrightarrow{i^*} \mathrm{Hom}(A,G)$$

exact

In our case, we have the full SES because C_n are all free.

$$\left(\text{so} \quad \begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[l, dashed, bend right] \arrow[r] & C \arrow[r] & 0 \end{tikzcd} \right)$$

So $H^n(X,A;G) := H^n(C^n(X,A;G), \delta)$.

Similarly, a SES of cochain complexes

$$\Big\Downarrow (*).$$

a LES of cohomology.

$$\cdots \to H^n(X,A) \to H^n(X) \to H^n(A) \xrightarrow{\delta} H^{n+1}(X,A) \to \cdots$$

why (*)? a cochain complex $(C^\bullet, \delta)$

gives a chain complex by $C_n := C^{-n}$, $n \in \mathbb{Z}$,

vice versa. (or $C_n := C^{N-n}$, $N \gg 0$).

Hence, anything true for chain complexes

is also true for cochain complexes

with grading inverted.

Similarly, we have excision and Mayer-Vietoris for H^n.

We can also define

- simplicial coho.
- cellular coho.

note: $X =$ a CW complex.

$$C_n^{CW}(X) = H_n(X^n, X^{n-1}).$$

$$C^n_{CW}(X;G) = ① \, H^n(X^n, X^{n-1}; G) \quad ?$$

$$\text{or } ② \, \mathrm{Hom}(H_n(X^n, X^{n-1}); G), \quad ?$$

Answer: They are the same.

$$H_n(X^n, X^{n-1}) = \mathbb{Z}^r. \qquad ② = \mathrm{Hom}(\mathbb{Z}^r, G) \cong G^r$$

$$① \underset{\substack{\uparrow \\ \text{excision}}}{\cong} \tilde{H}^n(X^n/X^{n-1}; G) = \tilde{H}^n(\bigvee_r S^n; G) \cong G^r.$$

Q: How are $H_n(X)$ and $H^n(X)$ related?

Guess: $H^n(X;G) = H^n(\mathrm{Hom}(C_\bullet(X);G))$

$$\overset{(?)}{=} \mathrm{Hom}(H_n(C_\bullet(X));G)$$

$$= \mathrm{Hom}(H_n(X);G).$$

(?) is not true, but close.

↳ universal coefficient thm.

Universal coefficient theorems. [algebra].

Suppose $C = (C_n, \partial)$ is a chain complex,

s.t. C_n is free $\forall n$.

(e.g. $C_n = C_n(X)$).

G = an ab. gp.

$\mathrm{Hom}(C; G)$ is a cochain complex.

Goal: Relate $H^n(C; G) := H^n(\mathrm{Hom}(C_n; G))$

and $\mathrm{Hom}(H_n(C); G)$.

Step 1: There is a natural map

$$h: H^n(C;G) \longrightarrow \mathrm{Hom}(H_n(C); G).$$

Let $Z_n := \ker \partial \le C_n$. $B_n := \mathrm{Im}\, \partial \le Z_n \le C_n$.

$[\varphi] \in H^n$ is represented by $\varphi \in \mathrm{Hom}(C_n; G)$

s.t. $\delta\varphi = 0 \iff \varphi\partial = 0$

$\iff \varphi = 0$ on $\mathrm{Im}\,\partial = B_n$

$\varphi|_{Z_n}$ induces a map $\underset{\overset{\shortparallel}{H_n}}{\dfrac{Z_n}{B_n}} \xrightarrow{\bar{\varphi}} G$.

<u>check</u>: $\varphi \longmapsto \bar{\varphi}$ gives a well-defined map

$$\begin{aligned} h : H^n(C; G) &\to \mathrm{Hom}(H_n; G) \\ [\varphi] &\longmapsto \bar{\varphi} \end{aligned}.$$

<u>Step 2</u>: h is surjective.

In fact, we will give a section

$$\mathrm{Hom}(H_n; G) \to H^n(C; G).$$

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] \arrow[l, dashed, bend left, "p"] & B_{n-1} \arrow[r] \arrow[l, dashed, bend left] & 0 \end{tikzcd}$$

SES splits since $B_{n-1} \subseteq C_{n-1}$ is free.

$\Rightarrow$ $\exists$ a projection $p : C_n \to Z_n$

$\Rightarrow$ every $\varphi_0 : Z_n \to G$ extends to C_n

$$\begin{tikzcd} Z_n \arrow[r, "\varphi_0"] & G \\ C_n \arrow[u, "p"] \arrow[ur, dashed, "\varphi := \varphi_0 p"'] & \end{tikzcd}$$

$\Rightarrow$ every $\bar{\varphi} \in \mathrm{Hom}(H_n ; G)$, $\bar{\varphi} : H_n = \frac{Z_n}{B_n} \to G$

$\rightsquigarrow$ $\varphi_0 : Z_n \to G$, $\varphi_0|_{B_n} = 0$.

$\rightsquigarrow$ $\varphi : C_n \to G$, $\varphi|_{B_n} = 0$

or $\delta\varphi = 0$.

$\rightsquigarrow$ $\varphi \in \ker\delta$.

$\Rightarrow$ we have $\mathrm{Hom}(H_n(C) ; G) \to \ker\delta \to H^n(C;G)$

$\bar{\varphi} \longmapsto \varphi$

check that $\bar{\varphi} \mapsto \varphi$ gives an inverse of

$h : \varphi \mapsto \bar{\varphi}$. $\Rightarrow$ h is onto.

Moreover, the SES

$$\begin{tikzcd} 0 \arrow[r] & \ker h \arrow[r] & H^n(C;G) \arrow[r, "h"] & \mathrm{Hom}(H_n;G) \arrow[r] \arrow[l, dashed, bend left, "\exists"] & 0 \end{tikzcd}$$

splits.

<u>Rmk</u>: However, the section of h depends on our choice of a section p which is not canonical.

So the SES splits with a noncanonical splitting.

$H^n(C;G) \cong \mathrm{Hom}(H_n;G) \oplus \ker h$

But the $\oplus$ is NOT canonical and somewhat arbitrary.

Step 3 : Analyze Ker h. (Homological algebra).

Spoiler : Ker h only depends on $H_n(C)$ and G.

$$\text{Ker } h = \text{Ext}(H_{n-1}(C), G)$$

How do we compute Ext ?

H = finitely generated ab. gp.

Then

- $\text{Ext}(H \oplus H'; G) = \text{Ext}(H, G) \oplus \text{Ext}(H', G)$
- $\text{Ext}(H, G) = 0$ if H is free
- $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$

In summary, we have:

Thm (Universal coefficient thm for H^n)

Suppose $C = (C_n, \partial)$ is a chain complex of free ab. gps.

Then $\exists$ a split SES:

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \overset{\leftarrow \cdots}{\to} Hom(H_n(C), G) \to 0$$

Moreover, the SES is natural.

Apply to $C_n = C_n(X)$, X = a space

Cor: $H^n(X;G) \cong Hom(H_n(X), G) \oplus Ext(H_{n-1}(X), G)$

Rmk: ① $H^n(X;G)$ is determined by $H_n(X)$ and $H_{n-1}(X)$.

② If $H_n(X)$ is free $\forall n$, then $H^n(X;G) = Hom(H_n(X), G)$.

Universal coefficient thm for H_n

Goal: Relate $H_n(C;G) = H_n(C \otimes G)$

and $H_n(C) \otimes G$.

Thm: (UCT for H_n)

Suppose $C = (C_n, \partial)$ is a chain complex of free abelian groups.

Then $\exists$ a natural SES

$$\begin{tikzcd} 0 \arrow[r] & H_n(C) \otimes G \arrow[r] & H_n(C;G) \arrow[r] & \mathrm{Tor}(H_{n-1}(C), G) \arrow[l, dashed, bend right] \arrow[r] & 0 \end{tikzcd}$$

The splitting is not natural.

Apply to $C_n = C_n(X)$.

properties of Tor:

- $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$
- $\mathrm{Tor}(\bigoplus A_i, B) \cong \bigoplus \mathrm{Tor}(A_i, B)$
- $\mathrm{Tor}(A, B) = 0$ if A or B is free.
- $\mathrm{Tor}(\mathbb{Z}_n, A) = \ker(A \xrightarrow{n} A)$.

Ext and Tor functors. (fast intro).

Ext is the derived functor of Hom:

A = ab. gp. (fin. gen.)

Choose$^{(*)}$ a free resolution of A:

$$\cdots \to F_1 \to F_0 \to A \to 0$$

apply $\text{Hom}(-, G)$: (removing last one).

$$\cdots \text{Hom}(F_1, G) \leftarrow \text{Hom}(F_0, G) \leftarrow 0$$

$$\text{Ext}^i(A, G) := H^i(\text{Hom}(\mathcal{F}, G)).$$

Rmk: $\text{Ext}^i(A,G)$ is independent of choice $(*)$.

Tor is the derived functor of $\otimes$

Choose a free resolution of A:

$$\cdots \to F_1 \to F_0 \to A \to 0$$

Apply $- \otimes G$:

$$\cdots \to F_1 \otimes G \to F_0 \otimes G \to A \otimes G \to 0$$

$$\mathrm{Tor}_i(A, G) := H_i(F \otimes G)$$

<u>Rmk</u>: In UCT's,

$\mathrm{Ext}(A,B)$ should be $\mathrm{Ext}^1(A,B)$

$\mathrm{Tor}(A,B)$ - - - - $\mathrm{Tor}_1(A,B)$.

Last time:

UCT $\Rightarrow$ $H^\bullet(X;G)$ is determined by $H_\bullet(X)$.
☹ and G.

Today: New structure on $H^\bullet$ ☺.

Cup product. (Hatcher 3.2).

X = a space
R = a ring (!)

$\Rightarrow$ $H^\bullet(X;R)$ is a ring.

We will give 2 approaches to the product structure.

1. via cochains (concrete)
2. via Kunneth formula (abstract).

1st approach.

(I) Cup product via cochains.

R = ring.

$$\varphi \in C^k(X;R) \quad , \quad \psi \in C^\ell(X;R)$$

Define their cup product

$$\varphi \cup \psi \in C^{k+\ell}(X;R) \quad \text{s.t.}$$

$$\forall \ \sigma : \Delta^{k+\ell} \longrightarrow X ,$$

$$(\varphi \cup \psi)(\sigma) := \varphi(\sigma|_{[v_0 \cdots v_k]}) \underset{\substack{\uparrow \\ \text{product in } R.}}{\cdot} \psi(\sigma|_{[v_k \cdots v_{k+\ell}]})$$

You check:

$$\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$$

"graded Leibniz rule".

$\Rightarrow$ cocycle $\cup$ cocycle = cocycle

cocycle $\cup$ coboundary = coboundary

$(\delta(\varphi \cup \psi) = \varphi \cup \delta\psi)$
if $\delta\varphi = 0$.

$\Rightarrow$ $C^k \times C^\ell \xrightarrow{\cup} C^{k+\ell}$

induces a well-defined product

$$H^k(X;R) \times H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R)$$

You check:
- associative, distributive.
- R-bilinear: $\forall r,s \in R$

  $(rx+sy) \cup z = r(x \cup z) + s(y \cup z)$
- identity $1 \in H^0(X;R)$

  $\hookrightarrow$ constant 1 function on all 0-simplices.

$\Rightarrow$ $H^\bullet(X;R)$ is a "graded commutative" R-algebra.
$\uparrow$ 1-word (if R commutative).

Rmk. We can similarly define cup product on $H^\bullet_\Delta(X;R)$ for X a Δ-complex. The isomorphism

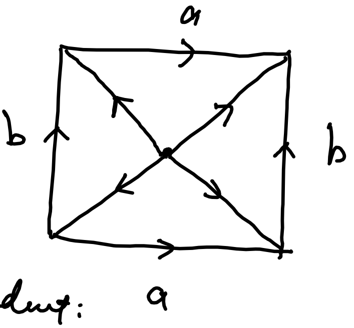
$$H^\bullet(X;R) \cong H^\bullet_\Delta(X;R)$$

is an isomorphism of $\cdot$ algebras.

Let's compute cup product using $H^\bullet_\Delta$!

Ex 1: $X = T^2$. $R = \mathbb{Z}$.

Δ-complex:



only interesting cup product:

$$H^1 \times H^1 \longrightarrow H^2.$$

UCT $\Rightarrow$ $H^1(T^2) \cong \mathrm{Hom}(\underset{\mathbb{Z}\{a,b\}}{\underset{\parallel}{H_1(T^2)}}, \mathbb{Z})$

$$\cong \mathbb{Z}\{\alpha, \beta\}.$$

$$\begin{aligned} \alpha : H_1 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 1 \\ b &\longmapsto 0 \end{aligned} \qquad \begin{aligned} \beta : H_1 &\longrightarrow \mathbb{Z} \\ a &\longmapsto 0 \\ b &\longmapsto 1 \end{aligned}$$

Need to find a simplicial 1-cocycle φ representing $\alpha \in H^1$.

$\in \mathrm{Hom}(C_1^\Delta, \mathbb{Z})$

$\Updownarrow$

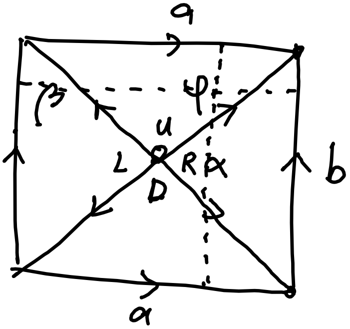
$$\varphi : C_1^\Delta = \mathbb{Z}^4 \oplus \mathbb{Z}\{a,b\} \longrightarrow \mathbb{Z}.$$

want: $\delta\varphi = 0$. "cocycle condition".

Define $\varphi : C_1 \to \mathbb{Z}$

s.t. ∀ edge σ

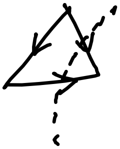
$$\varphi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ intersects } \cdots\alpha \\ 0 & \text{else.} \end{cases}$$



i.e. φ counts # times σ intersecting α.

You check $\delta\varphi = 0$.

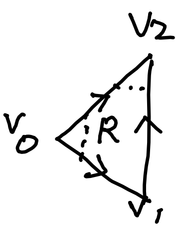
($\delta\varphi = 0$ since # times α enters and exits a face are equal with correct sign.)



$\varphi \cup \psi \in C^2 = \mathrm{Hom}(C_2, \mathbb{Z})$.

check: $\varphi \cup \psi$: $U, L, D \longmapsto 0$

$R \longmapsto 1$.



e.g.
$\varphi \cup \psi(R) = \varphi(R|_{v_0 v_1}) \cdot \psi(R|_{v_1 v_2}) = 1 \cdot 1 = 1$.

Finally, check

$F = D + R - U - L$ is a generator for $H_2(X) \cong \mathbb{Z}$

Conclude

$$H^1 \times H^1 \longrightarrow H^2 \text{ sends}$$

$$a^* \smile b^* = [\varphi] \smile [\psi] = [\varphi \smile \psi]$$

$$= [F] \text{ a generator for } H_2.$$

<u>You check</u>:

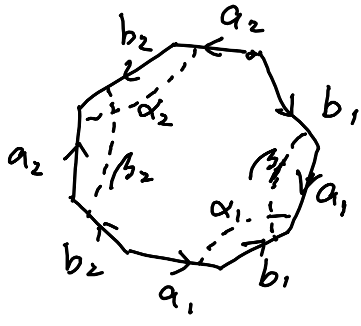
$$a^* \smile b^* = - b^* \smile a^*$$
$$a^* \smile a^* = 0$$

Hence,

$$H^\cdot(T^2) \cong \mathbb{Z}\langle a^*, b^*\rangle \Big/ \begin{matrix} (a^*)^2 = (b^*)^2 = 0 \\ a^* \smile b^* = -b^* \smile a^* \end{matrix}$$

$$= \Lambda(a^*, b^*)$$ → exterior algebra on 2 generators.

HW (generalize to Σ_g, $g \geq 1$).



Answer:

$$H^0(\Sigma_g) = \mathbb{Z}$$

$$H^1(\Sigma_g) = \mathbb{Z}\{\alpha_i, \beta_i\}_{i=1}^{g}$$

$$H^2(\Sigma_g) = \mathbb{Z}\{\gamma\}$$

where $\alpha_i \cup \alpha_j = 0$

$\beta_i \cup \beta_j = 0$

$$\alpha_i \cup \beta_j = -\beta_j \cup \alpha_i = \begin{cases} \gamma & i=j \\ 0 & i \neq j \end{cases}$$

observation:

nonzero cup product α_i, β_j

$\Longleftrightarrow$ α_i, β_j intersect.

$\leadsto$ Poincaré duality.

properties of cup product:

1. relative version: $A, B \subseteq X$. open subsets or subcomplexes.

$$H^k(X, A) \times H^\ell(X, B) \to H^{k+\ell}(X, A \cup B).$$

2. induced map is a ring map:

prop: If $f: X \to Y$

then $f^*: H^\cdot(Y; R) \to H^\cdot(X; R)$

is a ring homomorphism.

i.e. $f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$

Pf: Same holds for chain map $f^\#: C^\cdot \to C^\cdot$

$$(f^\# \varphi \cup f^\# \psi)(\sigma) = f^\# \varphi(\sigma|\_) \cdot f^\# \psi(\sigma|\ldots)$$

$$= \varphi f(\sigma|\ldots) \cdot \psi(f(\sigma|\ldots))$$

$$= (\varphi \cup \psi)(f\sigma)$$

$$= f^\#(\varphi \cup \psi(\sigma)).$$

$\square$

3. Graded commutativity.

<u>prop</u>: If R is a commutative ring, then $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$

$\forall \alpha \in H^k, \beta \in H^l$.

<u>Pf</u>. (straightforward but nontrivial)

By definition,

$(\varphi \cup \psi)(\sigma)$ and $(\psi \cup \varphi)(\sigma)$ only differ by a permutation of vertices on Δ^{k+l}.

Given $\sigma : [v_0 \cdots v_n] \to X$

define $\bar{\sigma} : [v_0 \cdots v_n] \xrightarrow{\tau} [v_n \cdots v_0] \xrightarrow{\sigma} X$

so $\bar{\sigma}(v_i) = \bar{\sigma}(v_{n-i}) \quad \forall i$

$\tau \in S_n$ is a product of $\frac{n(n+1)}{2}$ transpositions.

$\Rightarrow$ sign change by $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$

Define a map $\rho : C_n(X) \to C_n(X)$ $\forall n$.

by $\rho(\sigma) = \varepsilon_n \bar{\sigma}$.

<u>You check</u>: ① ρ is a chain map

i.e. $\partial\rho = \rho\partial$

② ρ is chain homotopic to id.

so $\rho_* = id$ on H_n.

[idea: Construct $P : C_n \longrightarrow C_{n+1}$

s.t. $\rho - id = \partial P + P\partial$].

$\rho \circlearrowright C_n \rightsquigarrow \rho^* \circlearrowright C^n$.

$$\rho^*(\psi \smile \varphi)(\sigma) = \varepsilon_{k+\ell}\, \psi(\sigma|_{[v_{k+\ell} \cdots v_k]}) \cdot \varphi(\sigma|_{[v_k \cdots v_0]})$$

$$(\rho^*\varphi \smile \rho^*\psi)(\sigma) = \varphi(\varepsilon_k \sigma|_{[v_k \cdots v_0]})\, \psi(\varepsilon_\ell \sigma|_{[v_{k+\ell} \cdots v_k]})$$

R commutative $\Rightarrow$

$$\varepsilon_{k+\ell}\, \rho^*(\psi \smile \varphi) = \varepsilon_k \varepsilon_\ell (\rho^*\varphi \smile \rho^*\psi)$$

① & ② $\Rightarrow$ $\rho^* = id$ on H^n.

$$\Rightarrow [\varphi] \cup [\psi] = \underset{\substack{\shortparallel \\ (-1)^{kl}}}{\frac{\varepsilon_{k+l}}{\varepsilon_k \cdot \varepsilon_l}} \cdot [\psi] \cup [\varphi].$$

$\square$.

$\boxed{\text{STOP}}$

<u>Rmk</u>: (1) $H^\bullet(X;R)$ is a graded commutative R-algebra

↳ 1 word.

i.e. $H^\bullet(X;R) = \bigoplus_k H^k(X;R)$

(2) $\alpha \in H^k(X;R)$ k odd

$$\Rightarrow \alpha \cup \alpha = -\alpha \cup \alpha \Rightarrow 2(\alpha \cup \alpha) = 0.$$

If ~~R~~ $H^{2k}(X;R)$ has no 2-torsion, then $\alpha \cup \alpha = 0$.
for $\alpha \in H^{\text{odd}}$.

<u>check</u> for $H^*(\Sigma_g;\mathbb{Z})$.

(3) Similar result holds for

$H^\bullet(X,A;R)$.

2nd approach to cup product (Kunneth).

X, Y spaces, R a ring.

Define cross product :

$$H^\bullet(X;R) \times H^\bullet(Y;R) \xrightarrow{\times} H^\bullet(X\times Y;R)$$

by $\alpha \times \beta = p_1^* \alpha \cup p_2^* \beta$. where
$$\begin{tikzcd} X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y. \\ X & \end{tikzcd}$$

Note : $\times$ is R-bilinear.

So induces a well-defined map on $\otimes_R$

$$H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \xrightarrow{\times} H^\bullet(X\times Y;R).$$
$$\alpha\otimes\beta \longmapsto \alpha\times\beta.$$

recall : $\otimes_R$ means $(rx)\otimes y = x\otimes(ry)$ $\forall r\in R$.

$\times$ is a map of graded R-modules.

Q: map of rings?

How to make $H^\bullet(X) \otimes H^\bullet(Y)$ a ring?

You check: Define

$$(a \otimes b) \cdot (c \otimes d) := (-1)^{|b||c|} (ac) \otimes (bd).$$

notation: if $b \in H^k$, then $|b| = k$. "degree"

Then the cross

$$H^\bullet(X;R) \otimes_R H^\bullet(Y;R) \to H^\bullet(X \times Y;R)$$

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Guess: $\times$ is an isomorphism.

Not true, but close.

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if X, Y are CW

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Ex:
- $R =$ any field. X, Y any CW. (Y finite cells in each dim).
- $R = \mathbb{Z}$.

  $Y = \Sigma_g, S^n, \mathbb{C}P^n, \dots$

  (non ex: $Y = \mathbb{R}P^2$, Lens spaces ...).

$\exists$ 2 approaches to pf of Thm:

2nd ~~1st~~ approach: algebraic version of Kunneth.

$\updownarrow$ ~~2nd~~ 1st approach: Eilenberg-Maclane axioms.

1st appr. sketch: Fix Y,

consider two functors. $\{\text{CW pairs}\} \to \text{ab. gp.}$

- $$h^n(X,A) := \bigoplus_{i+j=n} H^i(X,A;R) \underset{R}{\otimes} H^j(Y;R)$$
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Relative <u>version of Kunneth</u>:

$$H^\bullet(X,A) \otimes_R H^\bullet(Y,B) \xrightarrow[\cong]{\times} H^\bullet(X\times Y, A\times Y \cup X\times B)$$

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$$\tilde{H}^\bullet(X) \otimes_R \tilde{H}^\bullet(Y) \xrightarrow[\cong]{\times} \tilde{H}^\bullet(X\wedge Y)$$

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$\uparrow$ cup in $H^\bullet(X\times Y)$.

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and $H^k(Y;R)$ is a fini. gen. free R-module.

Hence, $H^n(X\times Y;R) \cong \bigoplus_{p+q=n} H^p(X;R)\otimes_R H^q(Y;R)$.

as R-modules,

Ex:

- R = any field. X, Y any CW.
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2nd approach to proving Kunneth $\boxed{\text{(algebra)}}$.

R = a commutative ring with 1.

Given chain complexes of R-modules:

(C, d) and (C', d')

Form a new chain complex $C \underset{R}{\otimes} C'$:

$$(C \underset{R}{\otimes} C')_n := \bigoplus_{p+q=n} C_p \underset{R}{\otimes} C_q .$$

$$D(c \otimes c')_n := dc \otimes c' + (-1)^{\deg c} c \otimes d c' .$$

Similarly, form $Hom_R(C, C')$:

$$Hom_R(C, C')_n := \prod_{q \in \mathbb{Z}} Hom_R(C_q, C'_{q+n})$$

$$D(f) := d'f - (-1)^n f d$$

$$\begin{tikzcd}
C_{q+1} \arrow[d, "d"'] \arrow[ddr, dashed] & \\
C_q \arrow[r, "f"] & C'_{q+n} \arrow[d, "d'"] \\
& C'_{q+n-1}
\end{tikzcd}$$

Thm: (Kunneth formula, algebraic version)

Suppose R is a PID.

and C, C' chain complexes of R-modules

s.t. C_n is a free R-module $\forall n$.

Then $\exists$ natural split SES's:

(T)

$$0 \to \bigoplus_{p \in \mathbb{Z}} H_p(C) \otimes H_{n-p}(C') \overset{\dashleftarrow}{\to} H_n(C \underset{R}{\otimes} C')$$

$$\to \bigoplus_{p \in \mathbb{Z}} \mathrm{Tor}_1^R(H_p(C), H_{n-p-1}(C')) \to 0.$$

and

(H)

$$0 \to \prod_{p \in \mathbb{Z}} \mathrm{Ext}_R^1(H_p(C), H_{p+n+1}(C')) \to H_n(\mathrm{Hom}_R(C, C'))$$

$$\overset{\dashleftarrow}{\to} \prod_{p \in \mathbb{Z}} \mathrm{Hom}_R(H_p(C), H_{p+n}(C')) \to 0.$$

Rmk: ① Take $R = \mathbb{Z}$.

$$C' : \cdots 0 \to 0 \xrightarrow{0} G \xrightarrow{0} 0 \to \cdots$$

Kunneth $\Rightarrow$ UCT for H_n.

② Same result holds for cochain complexes.

Set $C^n := C_{-n}$

$(C')^n := C'_{-n}$ $\quad \forall n \in \mathbb{Z}$.

For $(C \otimes C')^n$ a cochain complex.

$\Rightarrow$ Same formula as in (T) replacing $H_\bullet$ by $H^\bullet$.

③ Algebraic Kunneth $\Rightarrow$ Top. Kunneth.

$C^n := C^n(X;R)$

$(C')^n := C^n(Y;R)$

$H^k(Y;R)$ is fini. gen. free R-module

$\Rightarrow$ $\mathrm{Tor}_1^R(H^\bullet(X;R), H^\bullet(Y;R)) = 0$.

One can also define $\cup$ using $\times$:

For $X = Y$:

$$H^\bullet(X)\otimes H^\bullet(X) \xrightarrow{\times} H^\bullet(X\times X) \xrightarrow{\Delta^*} H^\bullet(X)$$

where $\Delta : X \hookrightarrow X\times X$ "diagonal."
$x \mapsto (x,x)$

composition is $\cup$.

<u>How about $H_\bullet$</u> ?

$$H_\bullet(X)\otimes H_\bullet(X) \xrightarrow[\text{not natural.}]{\times} H_\bullet(X\times X) \xleftarrow{\Delta_*} H_\bullet(X).$$

↑ wrong direction!

However, when $\times$ is an iso. (e.g. R = a field)

we can have:

$$H_\bullet(X) \xrightarrow{\Delta} H_\bullet(X)\otimes H_\bullet(X)$$

a "coproduct".

$\boxed{\text{STOP}}$

Example: $H^\bullet(T^2;\mathbb{Z}) = H^\bullet(S^1\times S^1;\mathbb{Z})$

$$\cong H^\bullet(S^1;\mathbb{Z}) \otimes H^\bullet(S^1;\mathbb{Z})$$

$$= \Lambda(a) \otimes \Lambda(b) \qquad |a|=|b|=1.$$

$$= \Lambda(a,b).$$

Example (HW):

$$H^\bullet(\mathbb{R}P^n;\mathbb{F}_2) \cong \mathbb{F}_2[\alpha]/(\alpha^{n+1}) \qquad \alpha\in H^1(\mathbb{R}P^n)$$

$$H^\bullet(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}) \qquad \alpha\in H^2(\mathbb{C}P^n).$$

Application:

Thm (Hopf):

If $\mathbb{R}^n$ has a division algebra structure over $\mathbb{R}$, (i.e. $ax=b$. $xa=b$ always solvable if $a\neq 0$)

then n is a power of 2.

e.g. $n = 1$ $\quad \mathbb{R}^1$

$n = 2$ $\quad \mathbb{R}^2 \cong \mathbb{C}$

$n = 4$ $\quad \mathbb{R}^4 \cong \mathbb{H}$ $\quad$ not commutative

$n = 8$ $\quad \mathbb{R}^8 \cong \mathbb{O}$ $\quad$ not associative

(Adams): Those are all.

Pf: Suppose $\mathbb{R}^n$ is a division algebra.

$$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\rightsquigarrow \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \xrightarrow{h} \mathbb{R}P^{n-1}.$$ "Hopf space".

$$\rightsquigarrow H^\bullet(\mathbb{R}P^{n-1}; \mathbb{F}_2) \xrightarrow{h^*} H^\bullet(\mathbb{R}P^{n-1}; \mathbb{F}_2) \otimes H^\bullet \cdots$$

$$\begin{tikzcd} H^\bullet(\mathbb{R}P^{n-1}; \mathbb{F}_2) \arrow[r, "h^*"] \arrow[d, phantom, "\shortparallel"] & H^\bullet(\mathbb{R}P^{n-1}; \mathbb{F}_2) \otimes H^\bullet \cdots \arrow[d, phantom, "\shortparallel"] \\ \mathbb{F}_2[\alpha]/\alpha^n & \mathbb{F}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n). \end{tikzcd}$$

Note inclusion $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}$

$$x \longmapsto (x, y_0).$$

and $$y \longmapsto (x_0, y)$$

sends $\alpha_1 \longmapsto \alpha$. $\quad \Rightarrow h^*(\alpha) = \alpha_1 + \alpha_2$.

$\alpha_2 \longmapsto \alpha$.

$\alpha^n = 0 \Rightarrow 0 = h^*(\alpha^n) = (\alpha_1 + \alpha_2)^n$

$$= \sum_k \binom{n}{k} \alpha_1^k \alpha_2^{n-k}$$

$\Rightarrow \binom{n}{k} = 0 \in \mathbb{F}_2 . \quad \forall\, 0 < k < n .$

$\Rightarrow$ n is a power of 2 $\square$.

Rmk: If we replace $\mathbb{R}$ by $\mathbb{C}$,

then we have $\binom{n}{k} \underset{\mathbb{Z}}{=} 0 \quad \forall\, 0<k<n$

$\Rightarrow n = 1$.

$\Rightarrow \exists$ no finite dim. divis. alg. over $\mathbb{C}$.

(∞-dim : $\mathbb{C}(X) = \{$ rational functions $\}$).

Example (Grassmann manifold).

$$G_n(\mathbb{C}^\infty) := \{ \mathbb{C}\text{-lines in } \mathbb{C}^\infty \}.$$

note:

$$G_1 = \mathbb{C}P^\infty.$$

There is a map

$$\begin{tikzcd}
(\mathbb{C}P^\infty)^{\times n} \arrow[r] & G_n(\mathbb{C}^\infty) & \\
(L_1, \dots, L_n) \arrow[r, mapsto] & \langle L_1, \dots, L_n \rangle \subseteq \mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty \arrow[r, phantom, "\overset{\shortparallel}{\mathbb{C}^\infty}."] & {}
\end{tikzcd}$$

invariant of S_n-action.

$$\begin{tikzcd}
H^*((\mathbb{C}P^\infty)^{\times n}) \arrow[r] \arrow[d, phantom, "\|s"] & H^*(G_n) \arrow[d, phantom, "\cup|"] \arrow[r, phantom, "\rightsquigarrow \text{Fact: } = !"] & {} \\
\mathbb{Z}[x_1, \dots, x_n] \arrow[r] & \mathbb{Z}[x_1 \dots x_n]^{S_n} &
\end{tikzcd}$$

$$\Rightarrow H^*(G_n) = \mathbb{Z}[x_1, \dots, x_n]^{S_n} \underset{\substack{\downarrow \\ \text{FTOSP}}}{=} \mathbb{Z}[c_1, \dots, c_n]$$

where $\prod_{i=1}^{n} (X - x_i) = \sum_{k=0}^{n} c_k(x_1, \dots x_n) \cdot X^k$.

Poincaré duality:

Focus on

$X := M$ a manifold

i.e. Hausdorff, second countable space s.t. every point has an open nbhd homeo to $\mathbb{R}^n$.

(n-dim).

Note: dim is intrinsic:

$\forall x \in M$,

$$\begin{aligned} H_i(M, M-x) &\cong H_i(\mathbb{R}^n, \mathbb{R}^n - 0) \\ &\cong \tilde{H}_{i-1}(\mathbb{R}^n - 0) \\ &\cong \tilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{else.} \end{cases} \end{aligned}$$

If M is compact, we also say it is a closed manifold to emphasize $\partial M = \phi$.

Ex: S^1, Σ_g, $\mathbb{R}P^n$, $\mathbb{C}P^n$, S^n,

lens spaces etc.

note: $\mathbb{C}P^n$. $x = [x_0 : \cdots : x_n] \in \mathbb{C}P^n$

$\exists\ i$ s.t. $x_i \neq 0$.

so $x = [\frac{x_0}{x_i} : \cdots : 1 : \cdots : \frac{x_n}{x_i}]$

Then $U := \{ [z_0 : \cdots : z_n] \mid z_i \neq 0 \}$

is a nbhd of x s.t. $U \cong \mathbb{C}^n = \mathbb{R}^{2n}$.

Thm (Poincaré duality).

For M^n a closed manifold

$$H_k(M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2) \quad \forall k.$$

If M^n is "orientable", then

$$H_k(M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z}) \quad \forall k.$$

Hence,

$$\dim H_K(M;\mathbb{Z}_2) = \dim H_{n-k}(M;\mathbb{Z}_2)$$

and if $H_k(M;\mathbb{Z})$ has no tusion

$$\operatorname{rank} H_k(M;\mathbb{Z}) = \operatorname{rank} H_{n-k}(M;\mathbb{Z}).$$

<u>Cor</u>: If M is odd-dimensional, then $\chi(M) = 0$

<u>Ex</u>: $M = T^n$. $H^*(M;\mathbb{Z}) = \Lambda(x_1, x_2, \cdots, x_n)$.

$$\operatorname{rank} H^k(M;\mathbb{Z}) = \operatorname{rank} \Lambda^k(x_1 \cdots x_n) = \binom{n}{k}.$$

$$\text{Thm} \Rightarrow \binom{n}{k} = \binom{n}{n-k}.$$

Intuition behind Poincaré duality:

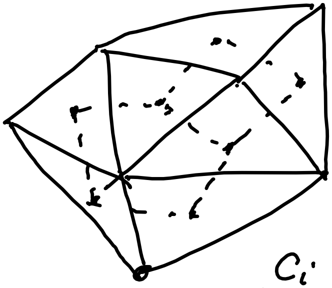
Say M is 2-dimensional.

with a Δ-complex structure:

$\rightsquigarrow$

a dual complex M'.

pic:



(say: dual is not a Δ-complex CW complex).

We have $\overset{= C_i}{C_i^{\Delta}(M)} \cong \overset{= C'_{n-i}}{C^{CW}_{n-i}(M')}$ $(n=2)$

note: over $\mathbb{Z}$, the sign might be ambiguous. (look at $i=1$).

over $\mathbb{F}_2$, no ambiguity.

[M orientable $\overset{(!)}{\Longleftrightarrow}$ it is possible to choose a sign convention to make it work].

You check:

$$\begin{tikzcd}
C_i \arrow[r, phantom, "\cong"] \arrow[d, "\partial"] & C'_{n-i} \arrow[d, "\partial'"] \\
C_{i-1} \arrow[r, phantom, "\cong"] & C'_{n-i+1}
\end{tikzcd}$$

$\partial' = \delta$

if we identify

$C^i(M') = C_i(M')^*$

$\tau^* \longleftrightarrow \tau$

$$\partial(\sigma) = \sum \sigma'$$

σ' is a face of σ.

$$\partial'(\tau) = \sum \tau'$$

τ is a face of τ'.

$$\left(\text{so} \quad \partial'(\tau) = \left(\delta(\tau^*)\right)^{*^{-1}} \right).$$

Hence,

$$H_i(M) = H_i(C, \partial) \overset{\text{duality map}}{\cong} H_{n-i}(C', \partial')$$

$$= H_{n-i}(C^\bullet(M'), \delta)$$

$$= H^{n-i}(M').$$

Classification of surfaces.

Thm 1: If M is a closed 2-manifold, then M is homeomorphic to one of

- Σ_g . $g \geq 0$ (orientable)
- $\Sigma_g \# \mathbb{R}P^2$, (nonorientable).

To prove Thm 1, we need

Thm 2 : (2-dim Poincaré conjecture).

For a closed 2-manifold M, TFAE:

① $\chi(M) = 2$.

② $M \cong S^2$

③ Every loop $\gamma : S^1 \hookrightarrow M$ separates M into 2 components. (Jordan curve).

Jordan curve thm

Pf: ② $\Rightarrow$ ③ . ② $\Rightarrow$ ① ✓

We will show ① $\Rightarrow$ ② and ③ $\Rightarrow$ ②.

assuming

Thm (Rado 1925): $M^2 \cong$ a simplicial complex K

$K^{(1)}$ is a connected graph.

Pick a maximal spanning tree $T \subseteq K^{(1)}$

note: $T \supseteq K^{(0)}$ ..

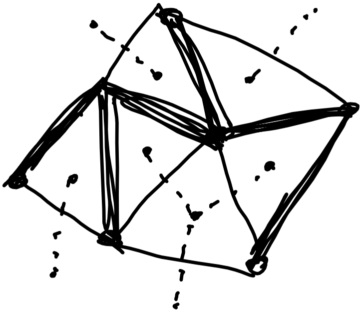
Build another graph $\Gamma \subseteq K$ as follows:

$V(\Gamma) =$ 2-simplices of K.

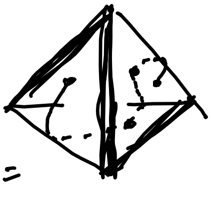
$E(\Gamma) = K^{(1)} - T$

pic:

$K^{(1)} =$



$K^{(1)} =$



T is a tree $\Rightarrow$ Γ is a connected graph.

$$\begin{aligned}
\chi(M) &= \chi(K) \\
&= V(K) - E(K) + F(K) \\
&= V(T) - \big(E(T) + E(\Gamma)\big) + V(\Gamma). \\
&= V(T) - E(T) + V(\Gamma) - E(\Gamma) \\
&= \chi(T) + \chi(\Gamma).
\end{aligned}$$

Lemma: For any finite connected graph G, $\chi(G) \leq 1$ with equality iff G = a tree.

Pf: You check.

① : $\chi(M) = 2 \Rightarrow \Gamma$ is a tree.

$\Rightarrow \Gamma$ has a nbhd $\cong \mathring{D}^2 \subseteq K$.

Thicken Γ and T to neighborhoods $\cong \mathring{D}^2$

s.t. $\overline{\text{nbhd}(\Gamma)} \cap \overline{\text{nbhd}(T)} \cong S^1$

Hence, $K \cong D^2 \underset{S^1}{\cup} D^2 \cong S^2$.

So ① $\Rightarrow$ ②.

Note ③ $\Rightarrow$ Γ is a tree $\Rightarrow$ ②

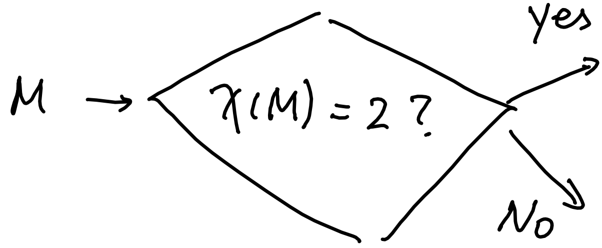
(if not, a loop in Γ will separate M and also separate $K^{(0)}$ into two subsets

But $\Gamma \cap T = \phi$, contradiction).

$\square$.

pf of Thm 1: Given M

we know $\chi(M) \leq 2$ by above.

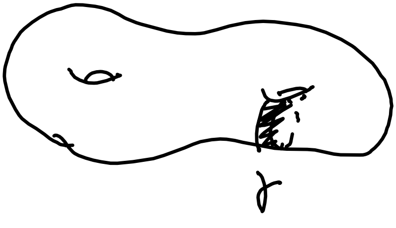


$M \cong S^2 = \Sigma_0$.

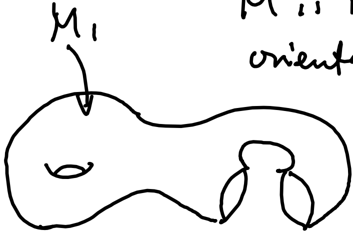
$\exists$ a loop γ in M s.t. $M \setminus \gamma$ is connected.

A neighborhood of γ in M has closure M_2 homeo to either $S^1 \times [0,1]$

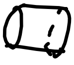
or Möbius band. (only when M is not orientable).

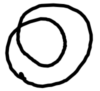


cut $\rightsquigarrow$



$M_1 := M - \mathring{M}_2$

$M_2 =$ 

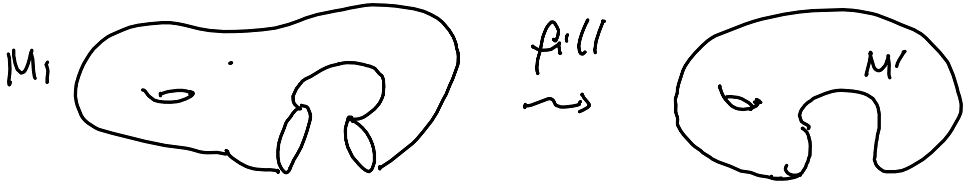
or $M_2 =$  ,

$\partial M_2 = S^1$.

Mayer-Vietoris

$$\Rightarrow \chi(M) = \chi(M_1) + \overset{0}{\overset{\shortparallel}{\chi(M_2)}} - \underset{= S^1 \sqcup S^1 \text{ or } S^1.}{\overset{0}{\overset{\shortparallel}{\chi(\partial M_2)}}}$$

Let $M' := M_1$ with ∂M_1 filled by disk(s).



MV ⇒

$$\chi(M') = \chi(M_1) + \underset{\text{or } \chi(D^2).}{\chi(D^2 \sqcup D^2)} - \underset{\overset{\shortparallel}{0}}{\chi(S^1 \sqcup S^1)}$$

Hence,

$$\chi(M') = \begin{cases} \chi(M) + 2 & \text{if } M_2 = \\ \chi(M) + 1 & \text{if } M_2 = \end{cases}$$

$$> \chi(M).$$

Repeat untill $\chi(M') = 2. \Rightarrow M' \cong S^2.$

In reverse, $M = S^2 \underset{g}{\#} (T^2)$ if orientable

or $M = S^2 \underset{k}{\#} (\mathbb{R}P^2) \underset{j}{\#} (T^2)$. □.

Next: Prove PD for any dim n in 3 steps:

(I) Orientations (Today)

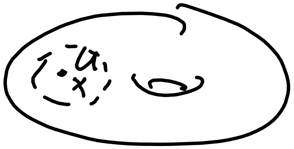
(II) Proof of PD. (Wed).

(III) Relate PD and cup product (Fri).

(I) Orientations.

connected

M^n = an n-manifold.



$\forall x \in M$, ← "local homology of M at x".

$$H_i(M, M-\{x\}) \underset{\text{excision}}{\cong} H_i(U, U-\{x\})$$

$$\cong H_i(\mathbb{R}^n, \mathbb{R}^n-\{0\})$$

$$\cong \tilde{H}_{i-1}(\mathbb{R}^n-\{0\})$$

$$\cong \tilde{H}_{i-1}(S^{n-1}) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{else.} \end{cases}$$

Notation: For $A \subseteq M$,

write $H_n(M/A) := H_n(M, M - A)$

"local homology of M at A"

Def: A local orientation of M at $x \in M$

is a choice of a generator

μ_x for $H_n(M/x) \cong \mathbb{Z}$

Let $\tilde{M} := \left\{ (x, \mu_x) \;\middle|\; x \in M,\ \mu_x \text{ is a local orientation at } x \right\}$

$\tilde{M} \longrightarrow M$ is a 2:1 covering map.

"orientation cover"

Def: M is orientable if $\tilde{M}$ is disconnected

An orientation of M is a choice of

a section $x \mapsto \mu_x$.

$$\begin{tikzcd} \tilde{M} \arrow[r] & M \arrow[l, bend right, dashed] \end{tikzcd}$$

Rmk: ① $\tilde{M} \to M$ gives a monodromy rep

$\pi_1(M) \xrightarrow{\rho} S_2 = \{\pm 1\}$.

M orientable iff ρ is trivial.

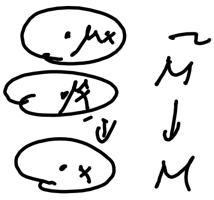
If $\nexists$ $\pi_1 M \twoheadrightarrow S_2$. then M is orientable.

(e.g. M = simply connected, S^n $n \geq 2$.

lens space L^m where $2 \nmid m$.).

$\pi_1 = \mathbb{Z}_m$.

② $\tilde{M}$ is orientable.

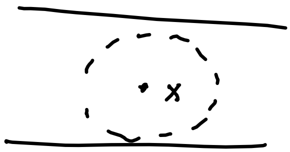


orientation given by $(x, \mu_x) \longmapsto \mu_x$.

Cor: Every nonorientable manifold is doubly covered by an orientable one.

Example: $M =$ Möbius band

a local orientation on M^2 is



equivalent to choosing a direction μ_x along a small circle around x.

check: If x moves along Möbius band, then $\mu_x \mapsto -\mu_x$.

$\Rightarrow$ The double cover $\tilde{M} \to M$ is nontrivial.
$\Rightarrow$ M not orientable

check: $\tilde{M} \cong S^1 \times [0,1]$. orientable.

Orientation

R = a commutative ring with 1.

(e.g. $R = \mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$).

M^n = an n-manifold.

$\forall x \in M$, $H_n(M/x ; R) \cong R$.

An R-orientation of M assigns to each $x \in M$

a generator $\mu_x \in H_n(M/x ; R) \cong R$

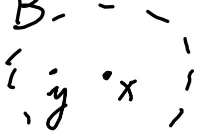
in a locally consistent way

i.e. $\forall x \in M$. $\exists$ open ball $B \ni x$

and $\exists$ $\mu_B \in H_n(M/B ; R)$ s.t.

$\forall y \in B$, $H_n(M/B) \longrightarrow H_n(M/y)$

$\mu_B \longmapsto \mu_y$.



<u>Recall</u>: $\mu \in R$ is a generator (of R as an R-module)

iff $R \cdot \mu = R$

iff μ is a unit in $R^\times$.

Define

$$M_R := \{ (x, \alpha_x) \mid x \in M, \ \underset{\substack{\uparrow \\ \text{any element.}}}{\alpha_x \in H_n(M \mid x ; R)} \}$$

$M_R \longrightarrow M$ is a cover of M with fiber $\overset{}{\underset{R.}{\cong}}$

An R-orientation of M

= a section $x \longmapsto \mu_x$ of $M_R \to M$

s.t. μ_x is a generator of $H_n(M/x ; R)$ $\forall x$.

Rmk: Last week, we discussed the case $R = \mathbb{Z}$.

$$\tilde{M} = \{ (x, \mu_x) \} \subseteq M_{\mathbb{Z}}.$$

$$H_n(M/x;R) \cong H_n(M/x;\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} R.$$

$\forall\, r \in R$,

$$M_r := \{ (x, \pm \mu_x \otimes r) \} \subseteq M_R.$$

where μ_x is a $\mathbb{Z}$-generator of $H_n(M/x;\mathbb{Z})$

If $r = -r$, then $M_r = M_{-r} \cong M$

If $r \neq -r$, then $M_r \cong \tilde{M}$

So $M_R = \bigcup_{r \in R} M_r$ disjoint union

except $M_r = M_{-r}$.

<u>Ex</u>: $R = \mathbb{Z}$.

$$M_{\mathbb{Z}} \cong \bigcup_{k=0}^{\infty} M_k \qquad M_0 = M. \quad M_1 = M_2 = \cdots = \tilde{M}.$$

Rmk: ① If M^n is orientable (i.e. $\mathbb{Z}$-orientable) then M^n is R-orientable $\forall R$.

(since $\{\pm 1\} \subseteq R^{\times}$.)

② If M^n is nonorientable (over $\mathbb{Z}$), then M is R-orientable iff $\exists\, r \in R^{\times}$ s.t. $2r = 0$.

(so $M_r \cong M$. an R-orientation is

$$x \longmapsto \pm \mu_x \otimes r \in H_n(M/x\,;R)$$

well-defined since $-r = +r$).

Hence, any M^n is $\mathbb{Z}_2$-orientable.

Most important cases : $R = \mathbb{Z}_2$ or $\mathbb{Z}$.

Thm: Let M^n be a closed connected manifold.

(a) If M is R-orientable, then the map

$$H_n(M;R) \longrightarrow H_n(M|x;R) \cong R$$

is an isomorphism $\forall x \in M$.

(b) If M is not R-ori. then $H_n(M;R) \to H_n(M|x;R) \cong R$ is injective with image $\cong \{r \in R \mid 2r = 0\}$.

(c) $H_i(M;R) = 0 \quad \forall i > n$.

Rmk. (a) $\Rightarrow H_n(M;R) \cong R$.

An orientation on M determines a choice of a generator $[M]$ of $H_n(M;R) \cong R$.

$\hookrightarrow$ "fundamental class"

Converse is also true. $\left(\underset{M}{H_n(M)} \xrightarrow{\cong} \underset{Mx}{H_n(M|x)} \right)$ $M \hookrightarrow Mx$

Lemma: M manifold, $A \subset M$ compact.

① If $x \mapsto \alpha_x$ is a section of $M_R \to M$.

then $\exists!\ \alpha_A \in H_n(M|A;R)$

s.t. $H_n(M|A;R) \xrightarrow{\text{res.}} H_n(M|x;R)$

$\alpha_A \longmapsto \alpha_x$.

② $H_i(M|A;R)=0 \quad \forall i>n$.

pf: (Lemma $\Rightarrow$ Thm). Take $A=M$ compact.

$H_i(M|M) = H_i(M,\emptyset) = H_i(M)$. (c)✓.

$\Gamma_R(M) :=$ { sections of
$$\begin{tikzcd} M_R \arrow[d] \\ M \arrow[u, bend right, dotted] \end{tikzcd}$$
} an R-module,

$\forall x \in M$, $H_n(M;R) \longrightarrow H_n(M|x;R)$

$\alpha \longmapsto \alpha_x$

The map $x \mapsto \alpha_x$ gives a section in $\Gamma_R(M)$.

$H_n(M;R) \longrightarrow \Gamma_R(M)$ is an R-module map.

$\alpha \longmapsto \alpha_x$

Lemma (1) says

$$\begin{tikzcd} H_n(M;R) \arrow[r] & \Gamma_R(M) \\ \alpha \arrow[r, mapsto] & (x \mapsto \alpha_x) \end{tikzcd}$$

is an isomorphism

M connected $\Rightarrow$ each section in $\Gamma_R(M)$ is uniquely determined by its value at one point $x \in M$.

(uniqueness of path lifting).

Hence,

$$\begin{tikzcd} H_n(M;R) \arrow[r, "\cong"] & \Gamma_R(M) \arrow[r, hook, "\mathrm{Res}"] & H_n(M|x;R) \cong R. \\ \alpha \arrow[r, mapsto] & (x \mapsto \alpha_x) \arrow[r, mapsto] & \alpha_x \end{tikzcd}$$

(a) If M is R-orientable,

then Res is an iso. since image contains a generator.

(b) M is not R-orientable.

$$M_R = \bigcup_{r\in R} M_r \quad \text{where } M_r \cong \begin{cases} \tilde{M} & \text{if } 2r \neq 0 \\ M & \text{if } 2r = 0 \end{cases}$$

Any section $M \to M_R$ is of the form

$x \longmapsto \pm\mu_x \otimes r$ for some $r \in R$ $\quad r = -r$.

So

$$H_n(M;R) \cong \Gamma_R(M) \xhookrightarrow{Res} H_n(M|x;R)$$

has image $= \{ r \in R \mid 2r = 0 \}$.

$\square$.

pf of Lemma

idea :

Step 1 : If lemma holds for compact subsets A, B, $A \cap B$. then it holds for $A \cup B$.

Mayer-Vietoris :

$$0 \to H_n(M|A\cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A\cap B)$$

$\wr$

$= H_{n+1}(M|A\cap B) = 0$ by Lemma ② for $A \cap B$.

Lemma ② for $A \cup B$ $\Leftarrow$ ② for $A, B, A\cap B$.

$\Phi(\alpha) = (\alpha, -\alpha)$. $\Psi(\alpha, \beta) = \alpha + \beta$.

Lemma ① for $A \cup B$:

<u>existence</u>: Suppose $x \mapsto \alpha_x$ is a section of $M_R \to M$.

① for $A, B, A \cap B$ gives:

$$\alpha_A, \quad \alpha_B, \quad \alpha_{A \cap B}.$$

Moreover, by uniqueness, both α_A, α_B restrict to $\alpha_{A \cap B}$.

so $(\alpha_A, -\alpha_B) \in \operatorname{Ker} \Psi = \operatorname{Im} \Phi$

$\Rightarrow \exists\ \alpha_{A \cup B}$ s.t. $\Phi(\alpha_{A \cup B}) = (\alpha_A, -\alpha_B)$. ✓

<u>uniqueness</u>:

note: if $\overset{\alpha_{A \cup B} - \alpha'_{A \cup B}}{\alpha} \in H_n(M|A \cup B)$ restricts to $\alpha_x = 0$ in $H_n(M|x)$ $\forall x \in A \cup B$, then same holds for its restrictions to $H_n(M|A)$ and $H_n(M|B)$, hence are zero by hypothesis.

$\Rightarrow \Phi(\alpha) = (\alpha, -\alpha) = 0$

$\Rightarrow \alpha = 0$ since Φ is injective.

Step 2: Reduce lemma to $M^n = \mathbb{R}^n$.

$A \subset M$ compact $\Rightarrow$ $A = A_1 \cup \cdots \cup A_m$

s.t. each A_i is compact

and $A_i \subseteq \mathbb{R}^n \subseteq M$.

Apply Step 1 to $A = A_1 \cup \cdots \cup A_{m-1}$

$B = A_m$.

$$A \cap B = \underbrace{(A_1 \cap A_m) \cup \cdots \cup (A_{m-1} \cap A_m)}_{(m-1) \text{ compact sets.}}$$

by induction on m, we just need to consider when $m = 1$.

$A \subseteq \mathbb{R}^n \subseteq M$.

Excision $\Rightarrow$ $H_i(M|A) \cong H_i(\mathbb{R}^n|A)$ $\forall i$.

Step 3: Assume $M = \mathbb{R}^n$.

If $A \subseteq \mathbb{R}^n$ is convex

i.e. $\forall\, x, y \in A, \quad tx + (1-t)y \in A$ $\forall\, t \in [0,1]$.

$\nearrow$ "deformation retracts"

then $A \rightsquigarrow x$ for any $x \in A$

$\mathbb{R}^n \setminus A \rightsquigarrow$ a sphere centered at x,

So $H_i(\mathbb{R}^n/A) \longrightarrow H_i(\mathbb{R}^n/x)$ is iso $\forall i$.

$$\alpha_A \longmapsto \alpha_x$$

$\rightsquigarrow$ exists uniquely.

lemma ✓ for A.

By induction, lemma holds for:

$$A = A_1 \cup \cdots \cup A_m$$

where A_i is compact convex

Finally, if $A \subseteq \mathbb{R}^n$ arbitrary compact set.

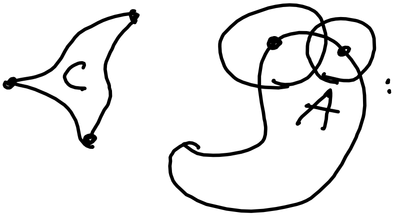
Take any $[z] \in H_i(\mathbb{R}^n/A)$

represented by $z \in C_i(\mathbb{R}^n)$ s.t.

$$\partial z \in C_{i-1}(\mathbb{R}^n - A).$$

say $\partial z = \sum_\alpha n_\alpha \sigma_\alpha$ $\quad \sigma_\alpha : \Delta^i \to \mathbb{R}^n - A$

let $C := \bigcup_\alpha \operatorname{im}(\sigma_\alpha)$ compact in $\mathbb{R}^n - A$.



Cover A by closed balls B_i's s.t. $B_i \cap C \neq \emptyset$

$K := \bigcup_i B_i$ a finite union of convex sets.
so lemma holds for K.

$C \subseteq \mathbb{R}^n - K. \Rightarrow z \in C_i(\mathbb{R}^n/K)$

so $\quad H_i(\mathbb{R}^n/K) \longrightarrow H_i(\mathbb{R}^n/A)$

$$[z] \longmapsto [z].$$

$[z]=0$ in $H_i(M|K)$

$\Rightarrow$ $[z]=0$ in $H_i(M|A)$

$\Rightarrow$ (2) ✓.

(1) existence: easy. Take any ball $B \supseteq A$.

$$H_n(\mathbb{R}^n|B) \longrightarrow H_n(\mathbb{R}^n|A)$$

$$\alpha_B \longmapsto \alpha_A.$$

uniqueness: Take $\alpha = \alpha_A - \alpha_A'$.

$$\begin{matrix}\alpha \\ \shortparallel \\ [z]\end{matrix}.$$

$$H_n(\mathbb{R}^n/K) \longrightarrow H_n(\mathbb{R}^n/A)$$

$$\alpha^K = [z] \longmapsto \alpha = [z]$$

assume: $\alpha_x = 0 \quad \forall x \in A$.

claim: $(\alpha^K)_x = 0 \quad \forall x \in K$.

since $K = \bigcup B_i$, $H_n(\mathbb{R}^n/B) \xrightarrow{\cong} H_n(\mathbb{R}^n/x)$. $\forall x \in B$.

Then $\alpha^K = 0$ by lemma (1) uniqness for K.

So $\alpha = 0$.

$\square$.

Thm (c) also holds for noncompact manifolds

prop: If M is a connected n-manifold, then $H_i(M;R)=0 \quad \forall i \geq n$.

pf: $\forall [z] \in H_i(M;R)$ represented by a cycle $z \in C_i(M;R)$.

z has compact image in M

Take open $U \subseteq M$ s.t. image $(z) \subseteq U$ and $\overline{U}$ is compact.

$V := M - \overline{U}$.

LES of the triple $(M, U \cup V, V)$:

$$H_{i+1}(M, U \cup V) \to H_i(U \cup V, V) \to H_i(M, V)$$

$i > n$: $H_{i+1}(M, U\cup V) = 0$, $H_i(M,V) = 0$

since $U \cup V$ and V have compact complement.
(Lema (2)).

$$\begin{tikzcd}
0 = H_i(U \cup V, V) \arrow[r] & H_i(M, V; R) = 0. \\
H_i(U) \arrow[u, phantom, "\cong \quad (MV)"] \arrow[r] & H_i(M) \arrow[u] \\
{[z]} \arrow[r, mapsto] \arrow[d, phantom, "\parallel"] & {[z]}. \\
0 &
\end{tikzcd}$$

$\Rightarrow$ z is a boundary in U

hence also in M.

$\Rightarrow$ $H_i(M; R) = 0$. $\square$.

Our goal is to prove:

PD: $H^k(M;R) \cong H_{n-k}(M;R)$.

What is the map?

Cap product:

$X =$ any space. $R =$ com. ring.

Define

$$C_k(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$$

by

$$\sigma \frown \varphi = \underbrace{\varphi(\sigma|_{[v_0 \cdots v_\ell]})}_{\ell\text{-simplex}} \cdot \underbrace{\sigma|_{[v_\ell \cdots v_k]}}_{k-\ell\text{-simplex.}}$$

You check: $\partial(\sigma \frown \varphi) = (-1)^\ell (\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$

We have a well-defined R-bilinear map:

$$H_K(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{K-\ell}(X;R)$$

$\exists$ relative versions.

Cap product is natural:

If $f: X \to Y$ is cont.

then

$$f_*(\alpha) \frown \varphi = f_*(\alpha \frown f^*(\varphi))$$

$\forall \alpha \in H_\bullet(X)$, $\varphi \in H^\bullet(Y)$

Thm (PD):

If M^n is closed R-oriented manifold with fundamental class $[M] \in H_n(M;R)$

then

$$\begin{aligned} D: H^k(M;R) &\longrightarrow H_{n-k}(M;R) \\ \alpha &\longmapsto [M] \frown \alpha \end{aligned}$$

is iso $\forall k$.

Recall:

(lemma) compact

orientable ⇒ ↠
connected ⇒ ↪.

$$\begin{tikzcd} H_n(M;R) \arrow[r, "\cong"] & \Gamma_R(M) \arrow[r, "\cong"] & H_n(M|x;R) \\ {[M]} \arrow[rr, mapsto] & & \text{a generator.} \end{tikzcd}$$

To prove Thm, we want to use an inductive Mayer-Vietoris argument again.

But open subsets are not compact.

We need a stronger version of PD for noncompact manifold.

Cohomology with compact support.

Define

singular cochain complex

$$C_c^i(X;G) \subseteq C^i(X;G) = \mathrm{Hom}(C_i(X), G).$$

$\varphi \in C_c^i(X;G)$ if $\exists$ a compact $K \subseteq X$ s.t. $\varphi = 0$ on all chains in $X \setminus K$.

note : $\delta\varphi \in C_c^{i+1}(X;G)$.

$$H_c^i(X;G) := H^i(C_c^\bullet(X;G), \delta)$$

"cohomology with compact support".

Rmk: ⓪ X compact $\Rightarrow H^\bullet = H_c^\bullet$.

① A proper map $X \xrightarrow{f} Y$ (i.e. f^{-1}(compact) is compact).

induces $H_c^\bullet(Y) \xrightarrow{f^*} H_c^\bullet(X)$

② $H_c^\bullet(X)$ is not homotopy invariant, but only properly homotopy invariant.

i.e. $X \times I \to Y$ proper.

Ex: $H_c^1(\mathbb{R}) \cong \mathbb{Z}$

$\neq$

$H_c^1(*) = H^1(*) = 0$.

Note: $K \hookrightarrow L$ compact subsets in X

$$\begin{tikzcd} K \arrow[r, hook] & L \\ {} \arrow[d, squiggly] & \\ C^i(X, X \setminus K) \arrow[r, hook] & C^i(X, X \setminus L) \end{tikzcd}$$

$$C_c^i(X) = \bigcup_{K \text{ compact}} C^i(X, X-K)$$

$$= \varinjlim_{K} C^i(X, X-K)$$

"directed limit".
or "colimit"

Suppose I is a poset s.t.

$\forall \alpha, \beta \in I$, $\exists \gamma \in I$, $\alpha \leq \gamma, \beta \leq \gamma$

Given a functor $I \longrightarrow Ab\,Gp$

$$\alpha \longmapsto G_\alpha$$

$$\alpha \leq \beta \longmapsto G_\alpha \xrightarrow{f_{\alpha\beta}} G_\beta$$

define

$$\varinjlim G_\alpha := \coprod_\alpha G_\alpha \Big/ \sim$$

where $a \sim b$ if

$$\begin{tikzcd} G_\alpha \arrow[dr, "f_{\alpha\gamma}"] & \\ & G_\gamma \\ G_\beta \arrow[ur, "f_{\beta\gamma}"'] & \end{tikzcd}$$

$a \mapsto$ same, $b \mapsto$

Ex: $I = \{ \text{compact } K \subseteq X \}$.
with inclusions.

[algebra].

Fact: Taking colimit preserves SES's.
hence commutes with homology.

$$H_c^i(X) \overset{\text{def}}{=} H^i(C_c^\bullet(X))$$

$$= H^i(\varinjlim_K C^\bullet(X, X-K))$$

$$\overset{*}{=} \varinjlim_K H^i(C^\bullet(X, X-K))$$

$$= \varinjlim_K H^i(X, X-K) .$$

Ex: $H_c^1(\mathbb{R}) = \varinjlim_{\text{Balls}} \underset{\underset{\mathbb{Z} \;\; \forall B}{\text{IIS.}}}{H^1(\mathbb{R}, \mathbb{R}-B)} \cong \mathbb{Z}$.

Duality for noncompact manifolds.

$M^n :=$ R-oriented n-manifold

$K \overset{i}{\hookrightarrow} L \hookrightarrow M$ gives:

$$\begin{tikzcd}
H_n(M|L) \times H^k(M|L) \arrow[d, "i_*"'] \arrow[rd, "\cap"] & \\
H_n(M|K) \times H^k(M|K) \arrow[u, "i^*"'] \arrow[r, "\cap"] & H_{n-k}(M)
\end{tikzcd}$$

Lemma last time:

$\exists!\ \mu_K \in H_n(M|K)$ that restricts to the chosen R-orientation $\mu_x \in H_n(M|x)$ $\forall x \in K$.

Same for $\mu_L \in H_n(M|L)$.

uniqueness $\Rightarrow$ $i_*(\mu_L) = \mu_K$.

naturality of $\cap$ $\Rightarrow$ $\mu_K \cap x = \mu_L \cap i^*(x)$

Hence,
$$H^k(M|K) \longrightarrow H_{n-k}(M)$$
$$x \longmapsto \mu_K \cap x$$
induces a map at the colimit:
$$D_M: H_c^k(M) \longrightarrow H_{n-k}(M)$$

Thm. (PD for noncompact M).
If M is R-oriented
then D_M is an iso.

Pf:
Step 1: For U, V open subsets in M.
if D_U, D_V, $D_{U\cap V}$ are iso's.
so is $D_{U\cup V}$.

Lemma: MV and D commute up to a sign:

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^k(U\cap V) \arrow[r, "(!)"] \arrow[d, "D_{U\cap V}"] & H_c^k(U)\oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H_c^k(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

(pf skipped):

Note (!): If $U \overset{i}{\hookrightarrow} X$ open subset

$$C_c^i(U) \xrightarrow{i_*} C_c^i(X)$$

$$\left(\varphi : C_i(U) \to R\right) \longmapsto \left(\varphi : C_i(X) \to R\right)$$

compactly supported $\qquad\qquad$ $\varphi := 0$ outside U.

Five Lemma + Lemma above $\Rightarrow$ step 1.

Step 2: Suppose $U_1 \subseteq U_2 \subseteq \cdots$ open s.t. D_{U_i} is iso $\forall i$

Let $M := \bigcup_i U_i$

D_M is also an iso.

$$\begin{tikzcd}
H_c^k(U_i) = \varinjlim\limits_{K \subseteq U_i} H^k(U_i/K) \arrow[d, phantom, "\cong", "\text{excision}"'] \\
H^k(M/K)
\end{tikzcd}$$

$$\begin{tikzcd}
\varinjlim\limits_{U_i} H_c^k(U_i) \arrow[d, "D_{U_i}"', "\cong"] & = \varinjlim\limits_{U_i} \varinjlim\limits_{K \subseteq U_i} H^k(M/K) = \varinjlim\limits_{K \subseteq M} H^k(M/K) = & H_c^k(M). \arrow[dl, "D_M \text{ is iso.}"] \\
\varinjlim\limits_{U_i} H_{n-k}(U_i) \arrow[r, phantom, "\cong"] & H_{n-k}(M) &
\end{tikzcd}$$

Step 3: Prove Thm for $M = \mathbb{R}^n$
(You).

Step 4: Prove Thm for $M = U \subseteq \mathbb{R}^n$ open.

$M = \bigcup_{i=0}^{\infty} U_i$ $\quad U_i$ convex open. Thm holds for U_i ✓

$V_i := \bigcup_{j \leq i} U_j$ finite union of U_j's.

$\therefore$ Step 1 $\Rightarrow$ Thm holds for V_i $\forall i$. ✓

Step 2 $\Rightarrow$ Thm holds for (induction).

$$M = \bigcup_{i=0}^{\infty} V_i \quad ✓$$

Step 5: Prove Thm for

$M =$ countable union of open sets in $\mathbb{R}^n$.

(Same as Step 4, replacing "convex open" by "open")

[Done for compact M].

Step 6: M = any manifold.

Step 2 + Zorn's lemma

$\Longrightarrow$ $\exists$ a maximal open $\mathcal{U} \subseteq M$ s.t. Thm holds.

Then $\mathcal{U}$ must be the entire M.

(if not, take $x \in M \setminus \mathcal{U}$.
x has a nbhd $V \cong \mathbb{R}^n$.
Thm holds for $\mathcal{U}$ and V
hence for $\mathcal{U} \cup V$. contradiction)

$\square$.

Algebraic Topology 1, Lecture 20

Friday, May 6, 2022 12:51 PM

Last time: Duality for noncompact manifolds

M^n = R- oriented manifold

(compact or not).

cohomology with compact support.

R = comm. ring.

$$H_c^i(M) \cong \varinjlim_{\substack{K \subseteq M \\ \text{compact}}} H^i(M|K)$$

$\rightsquigarrow$ $H^i(M, M-K)$

Lemma of last week: K compact

$\exists!\ \mu_K \in H_n(M|K)$ that restricts to the chose R- orientation $\mu_x \in H_n(M|x)$ $\forall x \in K$.

$$\rightsquigarrow \quad \begin{tikzcd} H^k(M|K) \arrow[r, "D_K"] & H_{n-k}(M) \\ \varphi \arrow[r, mapsto] & \mu_K \cap \varphi \end{tikzcd}$$

$\forall$ compact $K \subseteq M$.

↑ cap product.

induces a map at the colimit

$$H_c^k(M) \xrightarrow[]{\exists\ D_M} \ ''$$

$$\begin{tikzcd} H_c^k(M) \arrow[r, "\exists\ D_M", "\cong\ \cong"'] \arrow[d, phantom, "\cong" description] & H_{n-k}(M) \\ \xrightarrow[\substack{K\subseteq X \\ \text{compact}}]{\lim} H^k(M|K) \arrow[ur, "D_K"'] & \end{tikzcd}$$

$\nexists$ maximal $K=M$.

$\emptyset$ ↳ noncompact.

Rmk*: If M is compact, then

$\exists\ \overset{=[M]}{\mu_M} \in H_n(M/\overset{M=K}{M})$ in the limit s.t.

D_M is by $\varphi \longmapsto \mu_M \cap \varphi$

⇒ If M is not compact, $\overset{\rotatebox{90}{=}}{[M]}$ fundamental class.

$\nexists$ such μ_M since $H_n(M^n)=0$

(prop. last time).

Thm (PD for noncompact M)

If M^n is R-oriented,

then $H_c^k(M) \xrightarrow{D_M} H_{n-k}(M)$ is iso $\forall k$.

Pf. Step 1: For U, V open subsets in M

if $D_U, D_V, D_{U\cap V}$ are iso.

so is $D_{U\cup V}$.

$\cdots$ $* U\cup V$.

Lemma. MV and D commute up to sign:

MV for $H_c^\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^k(U\cap V) \arrow[r, "(!)"] \arrow[d, "D_{U\cap V}", "\cong"'] & H_c^k(U)\oplus H_c^k(V) \arrow[r] \arrow[d, "D_U \oplus -D_V", "\cong"'] & H_c^k(U\cup V) \arrow[r] \arrow[d, "D_{U\cup V}", "\cong"'] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

MV for $H_\bullet$

Five lemma.

pf skipped.

Rmk (!): ① $H_c^\bullet$ is covariant wrt open embeddings:

i.e. $U \overset{i}{\hookrightarrow} X$ open

$\rightsquigarrow$ $i_*: H_c^\bullet(U) \longrightarrow H_c^\bullet(X)$

② $H_c^\bullet$ is contravariant wrt proper maps:

i.e. $Z \xrightarrow{f} X$ proper.

$\rightsquigarrow$ $f^*: H_c^\bullet(X) \xrightarrow{f^*} H_c^\bullet(Z)$.

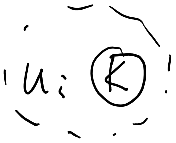
Step 2: Suppose $U_1 \subseteq U_2 \subseteq \cdots \subseteq \cdots$ one

Step 2: Suppose $U_1 \subseteq U_2 \subseteq \cdots \subseteq \cdots$ open s.t. D_{U_i} is iso. $\forall i$.

Let $M := \bigcup_{i=1}^{\infty} U_i$

D_M is also an iso. ✓

Recall:
$$H_c^k(U_i) = \varinjlim_{\substack{K:\, K \subseteq U_i \\ \text{compact}}} H^k(U_i/K) \cong H^k(M/K) \text{ by excision.}$$



$$\varinjlim_{U_i}\left(H_c^k(U_i)\right) = \underbrace{\varinjlim_{U_i} \varinjlim_{K \subseteq U_i}}_{=} H^k(M/K)$$

$$\begin{tikzcd}
\varinjlim_{U_i}\left(H_c^k(U_i)\right) \arrow[d, "D_{U_i}"'] \arrow[r, phantom, "\cong"] & \varinjlim_{K \subseteq M} H^k(M/K) = H_c^k(M) \arrow[dl, "D_M", "\cong"'] \\
H &
\end{tikzcd}$$

Step 3: Prove Thm for $M = \mathbb{R}^n$.

(You).

Step 4: Prove Thm for $M = U \subseteq \mathbb{R}^n$

↳ open.

$M = \bigcup_{i=0}^{\infty} U_i$ $\quad$ U_i ~~convex~~ open

Thm holds for U_i. ✓

$V_i := \bigcup_{j \le i} U_j$

$= U_1 \cup U_2 \cup \cdots \cup U_i$

convex.

Step 1 ⇒ Thm holds for V_i $\forall i$.

Step 2 ⇒ Thm holds for

$$M = \bigcup_{i=1}^{\infty} V_i \quad ✓.$$

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq \underset{\substack{\| \\ M.}}{\bigcup V_i}$$

Step 5: Prove Thm for

M = countable union of open

subsets in $\mathbb{R}^n$.

(Same as Step 4, replace "convex open" by "open").

(Thm holds for M compact).

Step 6: M noncompact. manifold.

Step 2 + "Zorn's lemma."

$\Rightarrow$ $\exists$ a maximal open subset $U \subseteq M$ s.t. Thm holds for U.

Then U must be the entire M.

$\square$

"Mayer - Vietoris argument".

Cup v.s Cap products. $\cup$ vs. $\cap$

You check:

$$\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha) \quad \text{------} \ (*)$$

$$\underline{\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)} \quad \text{------} \quad (*)$$

$\forall\ \alpha \in C_{k+l}$, $\varphi \in C^k$, $\psi \in C^l$.

Same if we pass ~~from~~ to H_{k+l}, H^k, H^l.

Fix $\varphi \in H^k$, we have:

$(H^l = H^l(X;R))$

$$\begin{tikzcd}
H^l \arrow[r, "h"] \arrow[d, "\varphi\smile"] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_R(H_l, R) \arrow[d, "(\frown\varphi)^*"] \\
H^{l+k} \arrow[r, "h"] & \mathrm{Hom}_R(H_{k+l}, R)
\end{tikzcd}$$

$(*) \Rightarrow$ diagram commutes.

When h are isomorphisms

(e.g. $R =$ a field,

or $R = \mathbb{Z}$ $\quad H_\bullet(X;\mathbb{Z})$ is free)

Then $\smile$ and $\frown$ determine each other,

and are dual under h.

cup product $\overset{\text{dual under } h.}{\longleftrightarrow}$ cap product

Last time: PD via $\cap$.

Today: PD via $\cup$.

Cup product and Poincaré duality

☆ Poincaré pairing: M = R-oriented n-manifold compact.

$$H^k(M;R) \times H^{n-k}(M;R) \xrightarrow{\langle , \rangle} R$$

$$(\varphi, \psi) \longmapsto (\underset{\in H^n}{\varphi \cup \psi})\underset{\in H_n}{[M]} \in R.$$

↑ fundamental class in $H_n(M)$.

(Algebra)

Def: A bilinear pai

$\Gamma n(M)$.

<u>Def</u>: A bilinear pairing

$$A \underset{R}{\otimes} B \xrightarrow{\langle , \rangle} R$$

is <u>nonsingular</u> if the induced maps

$$A \xrightarrow{a \mapsto (b \mapsto \langle a, b \rangle)} \mathrm{Hom}_R(B, R) = B^* \quad \text{dual of } B$$

$$B \xrightarrow{b \mapsto (a \mapsto \langle a, b \rangle)} \mathrm{Hom}_R(A, R) = A^*$$

are isomorphisms.

<u>Prop</u>: If M = closed R-oriented, R = a field or when $R = \mathbb{Z}$ and we consider

$$H^{\circ}(M; \mathbb{Z}) / \text{torsion}$$

then the Poincaré pairing is nonsingular.

Pf:

$$\begin{tikzcd}
\underset{\text{free}}{H^{n-k}(M)} \arrow[r, "h\ \boxed{\cong}"] & \mathrm{Hom}_R(\underbrace{H_{n-k}(M)}_{}^{\text{free}};R) \arrow[r, "D^* \cong"] & \mathrm{Hom}_R(\underbrace{H^k(M)}_{},R) \\
\text{UCT} \arrow[u] & \text{PD} \arrow[u] &
\end{tikzcd}$$

$$\psi \xmapsto{\cong} \left(\varphi \mapsto \psi([M]\cap\varphi)\right)$$

$$\| (*)$$

$$(\varphi \cup \psi)[M]$$

$$\|$$

$$\langle \varphi, \psi \rangle$$

Poincaré pairing.

PD $\Rightarrow$ D^* is iso

$\Rightarrow$ $\langle\ ,\ \rangle$ is nonsingular.

$\square$.

Take $\alpha \in \underbrace{H^k(M;\mathbb{Z})}_{}$

if α generates an infinite cyclic summand

(i.e. $k\alpha \neq 0 \ \forall k \in \mathbb{Z}$!

$\alpha \neq m\alpha'$ for $m \neq \pm 1$)

$H^k(M;\mathbb{Z})$

$$H^k(M;\mathbb{Z}) = \mathbb{Z}^r = \mathbb{Z}^{r-1} \oplus \mathbb{Z}$$

$$\hookrightarrow \langle \alpha \rangle .$$

$\Leftrightarrow$ $\exists\, \varphi \in \mathrm{Hom}(H^k, \mathbb{Z})$ s.t. $\varphi(\alpha) = \pm 1$.

[$\langle\, ,\, \rangle$ nonsingular $\Rightarrow$ $H^{n-k} \xrightarrow{\cong} \mathrm{Hom}(H^k, \mathbb{Z})$

$\exists!\, \beta \longmapsto \varphi$]

$\Leftrightarrow$ $\exists\, \beta \in H^{n-k}$ s.t. $\varphi(\alpha) = \langle \alpha, \beta \rangle = \pm 1$.

$$\langle \alpha, \beta \rangle = (\alpha \cup \beta)[M].$$

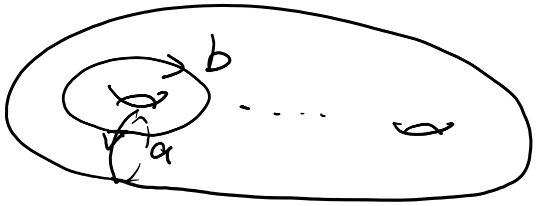
$\Leftrightarrow$ $\exists\, \beta \in H^{n-k}$ s.t. $\alpha \cup \beta$ generates $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$.

Cor ☆: $\alpha \in H^k$ generates an infinite cyclic summand

iff

if $\exists \beta \in H^{n-k}$ s.t. $\alpha \cup \beta$ generates $H^n(M;\mathbb{Z})$.

<u>Ex 1</u>: $M = \Sigma_g$.



$$H^1 \xrightarrow{\cong} \mathrm{Hom}(H_1, \mathbb{Z}).$$
$$\alpha \longmapsto a$$
$$\beta \longmapsto b$$

Then $\alpha \cup \beta$ generates H^2.

The existence of β is nontrivial.

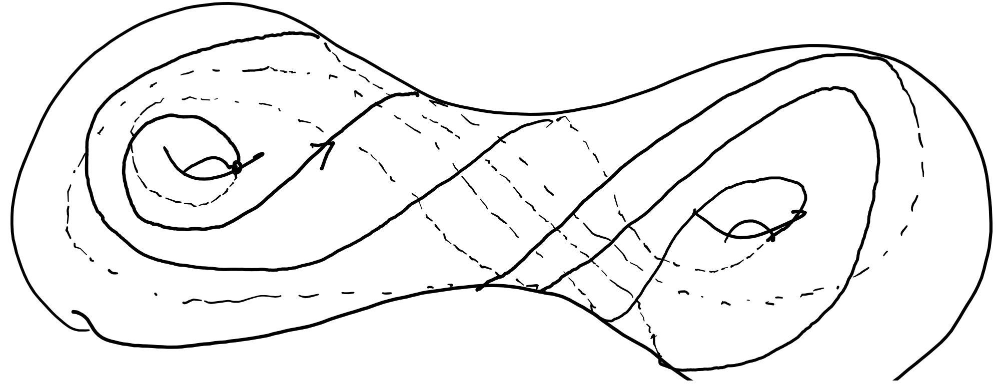
Take $\alpha = a^*$

where $a = \quad \overset{ma'}{\overset{\shortparallel}{a}} \in H_1(\Sigma_2;\mathbb{Z}). = \mathbb{Z}^4.$

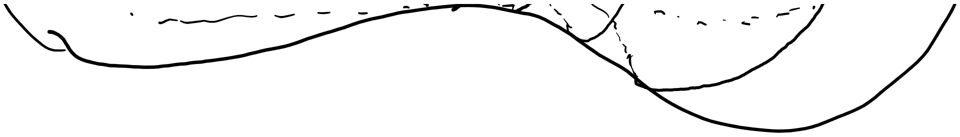
$\overset{?}{\exists}\beta$

$\alpha \cup \beta = \pm M$

$\overset{?}{H^2}$.



$H^{\overset{p}{2}}$.



We can ~~co~~ compute cup product on $H^1(\Sigma_g;\mathbb{Z})$ using Poincaré Pairing.

$$\underline{H^{1 \nearrow 2} \times H^{1 \nearrow 2}} \xrightarrow{\langle , \rangle} \mathbb{Z} \quad \text{nonsingular.}$$

$\langle , \rangle$ is ~~anti~~symmetric !

$$\langle \alpha, \beta \rangle = (\alpha \smile \beta)[\Sigma_g]$$

$$= (-1)^{\textcircled{1\cdot 1}} (\beta \smile \alpha)[\Sigma_g] = -\langle \beta, \alpha \rangle.$$

$H^1 \cong \mathbb{Z}^{2g}$

An inductive argument gives a $\mathbb{Z}$-basis $\{\alpha_i, \beta_i\}_{i=1,\dots,g}$ s.t.

$\langle \alpha_i, \alpha_j \rangle = 0$

$\langle \alpha_i, \beta_j \rangle = 0$ $\quad$ $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$

$\langle \beta_i, \beta_j \rangle = 0$.

(Fact from linear algebra):

Every nonsingular antisymmetric matrix is equivalent by a change of basis to

$$\Omega = \left.\begin{bmatrix} \begin{matrix}0&1\\-1&0\end{matrix} & & & 0 \\ & \begin{matrix}0&1\\-1&0\end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix}0&1\\-1&0\end{matrix} \end{bmatrix}\right\} 2g.$$

Ex 2: Compute cup product on $H^*(\mathbb{C}P^n;\mathbb{Z})$. using Poincaré.

$\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces iso on H^i_{CW} $\forall i \le 2n-2$.

inductively, assume

$H^{2i}(\mathbb{C}P^n)$ is generated by α^i $\forall i < n$.

$\alpha \in H^2$.

WTS: $H^{2n}(\mathbb{C}P^n)$ - - - - - - - - by α^n.

By Cor, $\exists\, m \in \mathbb{Z}$ s.t.

$$\alpha \cup (\underbrace{m\alpha^{n-1}}_{\beta}) = m(\alpha^n) \text{ generates } H^{2n}(\mathbb{C}P^n).$$

$\Rightarrow m = \pm 1$.

$\Rightarrow H^{\bullet}(\mathbb{C}P^n) = \mathbb{Z}[\alpha] / \alpha^{n+1}=0$. $\square$.

Ex 3: M^4 = a oriented closed topological 4-manifold.

$\langle , \rangle$ is symmetric on H^2

Thm (Freedman 1982).

A simply connected M^4 is classified by

- Poincaré pairing $\langle , \rangle$ on $H^2(M^4)$

  $\uparrow$ free abelian.

- $k(M) \in \{0,1\}$

  "Kirby - Siebenment invariant".

Cup product vs intersection product

M^n = closed oriented smooth manifold.

$A, B \subseteq M^n$ closed oriented smooth submanifolds of dim.

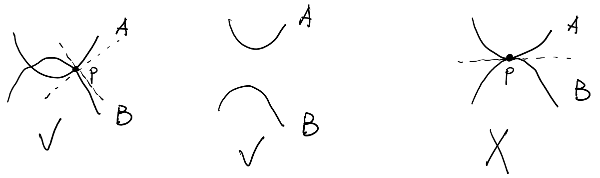
dim $A = n-i$
dim $B = n-j$

Assume A and B intersect transversely. ($A \pitchfork B$)

i.e. $\forall p \in A\cap B$, $T_pA + T_pB \longrightarrow T_pM$ is surjective.

tangent space

(Ex: $n=2$ $i=j=1$.



)

If $A \pitchfork B$, then $A\cap B$ is a submanifold of dim $= n-(i+j)$.
with SES:

$$0 \to T_p(A\cap B) \to T_pA \oplus T_pB \longrightarrow T_pM \to 0.$$

Rmk: orientations on (A, B, M) ⇝ orientation on $A\cap B$.

Poincaré duality:

$$D_M: H^i(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M;\mathbb{Z}).$$

$$\alpha \longmapsto [M]\cap\alpha$$

$$[A]^* = D_M^{-1}([A]) \longleftarrow [A]$$

$A \xhookrightarrow{inc} M$ gives: $H_{n-i}(A) \xrightarrow{inc_*} H_{n-i}(M)$

$$[A] \longmapsto inc_*[A] \quad (\text{noted as } [A]).$$

let $[A]^* := D_M^{-1}([A]) \in H^i(M;\mathbb{Z})$. dim $A = n-i$

Similarly, $[B]^* \in H^j(M;\mathbb{Z})$.

$[A\cap B]^* \in H^{i+j}(M;\mathbb{Z})$.

Thm☆ Suppose M is a closed oriented smooth manifold
A, B are . . . . submanifolds
with transverse intersection,
Then

$$\underset{\substack{\cap \\ H^i}}{[A]^*} \cup \underset{\substack{\cap \\ H^j}}{[B]^*} = \underset{\substack{\cap \\ H^{i+j}}}{[A\cap B]^*}.$$

i.e. "cup product is Poincaré dual to transverse intersection!"

pf: Next semester.

Rmk: ① orientation on $A\cap B$ = ± orientation $B\cap A$.

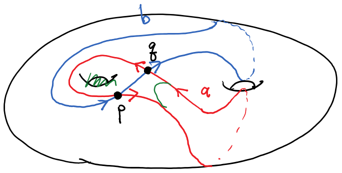
② In general, not every homology class in M can be represented by the fundamental class of a submanifold.

Thm (Thom): Every $x \in H_k(M, \mathbb{Z})$ has some integral multiple $nx = [A]$ for some submanifold $A \subseteq M$.

"Hodge conjecture:" asks for an algebraic analog for complex projective varieties.

Application of intersection product: (for computing cup product).

① $H^{\cdot}(\Sigma_g ; \mathbb{Z})$. cup product:



$[a], [b] \in H_1(\Sigma_2 ; \mathbb{Z})$

$[a]^* \cup [b]^* = [a \cap b]^* \in H^2(\Sigma_2 ; \mathbb{Z})$. $\overset{?}{=} 0$

$H_0 \ni [a \cap b] = \pm (\underset{H_0}{\overset{\uparrow}{[p]}} - \underset{H_0}{\overset{\uparrow}{[q]}}) = 0$ in H_0.

$\Rightarrow$ $[a]^* \cup [b]^* = 0$.

② $H^{\bullet}(\mathbb{C}P^n ; \mathbb{Z})$ cup product.

$h_i := \{ [x_0 : \cdots : x_n] \mid x_i = 0 \} \cong \mathbb{C}P^{n-1} \longrightarrow \mathbb{C}P^n$ "projective hyperplane".

$[h_i] \in H_{2n-2}(\mathbb{C}P^n)$.

$\longrightarrow \boxed{\tau_i := [h_i]^* \in H^2(\mathbb{C}P^n).}$

note: $\forall i, j$. $\tau_i = \tau_j$ in H^2. (writen as $\tau_i = \tau$) hyperplane class.

Pf: $\exists g \in \underset{\uparrow}{Gl_{n+1}(\mathbb{C})}$ s.t. $g^* \tau_i = \tau_j$.

$\mathbb{C}P^n \xrightarrow{g} \mathbb{C}P^n$.

$Gl_{n+1}(\mathbb{C})$ is connected $\Rightarrow$ $g \simeq id$. $\Rightarrow \tau_j = g^* \tau_i = id^* \tau_i = \tau_i$.

$\tau^n = \tau_1 \cup \tau_2 \cup \cdots \cup \tau_n$

$= [h_1]^* \cup [h_2]^* \cup [h_3]^* \cdots \cup [h_n]^*$

$\overset{\text{Thm.}}{=} [\underbrace{h_1 \cap h_2 \cap \cdots \cap h_n}_{}]^*$

$=$ a single point $\{ [1 : 0 : 0 : \cdots : 0] \}$.

$[pt]$ generates $H_0(\mathbb{C}P^n) \cong \mathbb{Z}$

$[pt]^*$ generates $H^{2n}(\mathbb{C}P^n) \cong \mathbb{Z}$

$\Rightarrow$ $H^{\cdot}(\mathbb{C}P^n) = \mathbb{Z}[\tau] / \tau^{n+1} = 0$

③ Bezout's Theorem in algebraic geometry.

Suppose $F = F(x_0, x_1, x_2)$ is homogeneous

$V = \{ [x_0 : x_1 : x_2] \in \mathbb{C}P^2 \mid F = 0 \} \subseteq \mathbb{C}P^2$.

an algebraic curve of degree $d = \deg F$

$V = \{ [x_0 : x_1 : x_2] \in \mathbb{C}P^2 \mid F = 0 \} \subseteq \mathbb{C}P^2.$

an algebraic curve of degree $d = \deg F$.

$\dim_{\mathbb{C}} = 1$ $\dim_{\mathbb{R}} = 2$

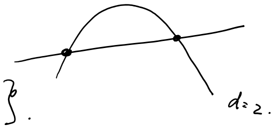
e.g. $F(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3$.

Q: what is $[V] \in \underset{\mathbb{Z}.}{\underset{\cong}{H_2(\mathbb{C}P^2)}}$?

$$\begin{tikzcd} {[V]} \arrow[d, mapsto] & \in H_2(V) \arrow[d] \\ ? = [V] & \in H_2(\mathbb{C}P^2). \end{tikzcd}$$

Pick a hyperplane $h \subseteq \mathbb{C}P^2$, s.t. $h \pitchfork V$.

claim: $h \cap V = \{ d \text{ points} \}$.



Pf: Say $h = \{ [x_0 : x_1 : x_2] \mid a x_0 + b x_1 + c x_2 = 0 \}$.

say $c \neq 0$. $x_2 = -\frac{a x_0 + b x_1}{c}$.

$h \cap V = \{ [x_0 : x_1 : -\frac{a x_0 + b x_1}{c}] \mid F(x_0, x_1, x_2(x_0, x_1)) = 0 \}$.

say $x_0 \neq 0$.

$= \{ [1 : x_1 : x_2(1, x_1)] \mid F(1, x_1, x_2(1, x_1)) = 0 \}$.

↳ poly. in 1-variable of deg $= d$.

$\Rightarrow$ d roots.

$= \{ d \text{ points} \}$.

Hence, $[V]^* \cup \underset{\tau \in H^2(\mathbb{C}P^2).}{\underset{\parallel}{[h]^*}} \overset{(\text{Thm}).}{=} [h \cap V]^* = d \underset{\tau^2 \in H^4(\mathbb{C}P^2)}{\underset{\parallel}{[\text{point}]^*}}$

$\Rightarrow$ $\boxed{[V]^* = d \cdot \tau.}$ in $H^2(\mathbb{C}P^2)$.

nonlinear → ← linear.

Given two homogeneous polynomials F_1, F_2.

$i = 1, 2$. say $\deg F_i = d_i$, $V_i = \{ F_i = 0 \} \subseteq \mathbb{C}P^2$.

Then $[V_1 \cap V_2]^* \overset{(\text{Thm}).}{=} \underset{d_1 \tau}{\underset{\parallel}{[V_1]^*}} \cup \underset{d_2 \tau}{\underset{\parallel}{[V_2]^*}} = d_1 d_2 \cdot (\tau^2) \in H^4(\mathbb{C}P^2)$.

Hence we obtain:

Bezout's Thm: If $\underset{d_1}{V_1} \pitchfork \underset{d_2}{V_2}$, then $V_1 \cap V_2 = \{ d_1 d_2 \text{ points} \}$.

Summary:

last time: cup product $\xleftrightarrow{h\text{-dual}}$ cap product

$h : H^k \longrightarrow \mathrm{Hom}(H_k ; R)$. (UCT).

Today: cup product $\xleftrightarrow{\text{Poincaré dual}}$ transverse intersection (☆)

$D: H^k \longrightarrow H_{n-k}$. (Poincaré duality)

Homotopy Groups

X = a space, $x_0 \in X$ base point. $I = [0,1]$.

" homotopy group "

Def: $\pi_n(X, x_0) := \{ f: (I^n, \partial I^n) \to (X, x_0) \} \Big/$ homotopy rel ∂I^n
i.e. $f_t(\partial I^n) = x_0 \ \forall t$.

$n=1$. $\pi_1(X, x_0) =$ fundamental group.

$n=0$: $I^0 = pt$. $\partial I^0 = \phi$.

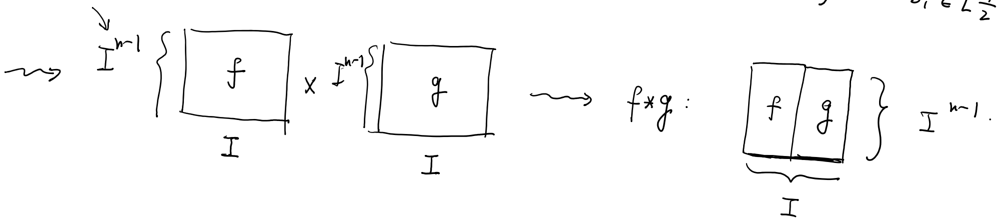
$\pi_0(X, x_0) \overset{\simeq}{\longleftrightarrow} \{$ path components of $X \}$. Not $\wedge$ a group!
necessarily.

$\pi_n(X, x_0)$ is a group when $n > 0$:

Given $f, g \in \pi_n$, $f, g : I \times I^{n-1} \longrightarrow X$

$$\rightsquigarrow (f*g)(s_1, s_2, \dots, s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \dots, s_n) & s_1 \in [\frac{1}{2}, 1]. \end{cases}$$

$f*g: I^n \longrightarrow X$.

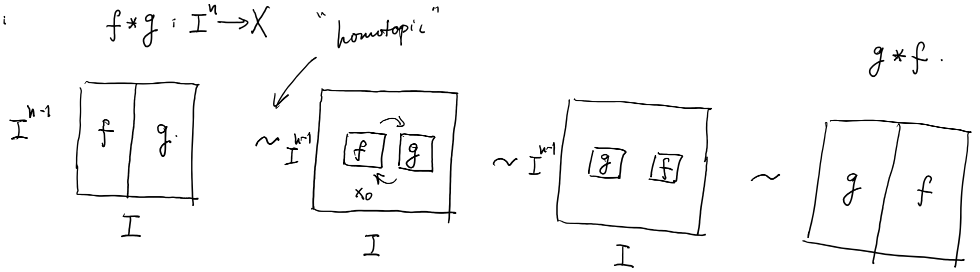


well-defined since $f|_{\partial I^n} = g|_{\partial I^n} = \{x_0\}$.

Rmk. π_n is a group $\uparrow$ (pf same as π_1).
for $n > 0$.

Prop: $\pi_n(X, x_0)$ is abelian when $n \geq 2$.

Pf: $f*g : I^n \to X$



$\cap$
$Map(I^n, X)$

□

Note: $Map((I^n, \partial I^n), (X, x_0)) \overset{\simeq}{\longleftrightarrow} Map((I^n/_{\partial I^n}, \partial I^n/_{\partial I^n}), (X, x_0))$.

||s.

$/\partial I^n, \ /\partial I^n), \ (X, x_0)).$

$\cong$

$(S^n, \ s_0)$

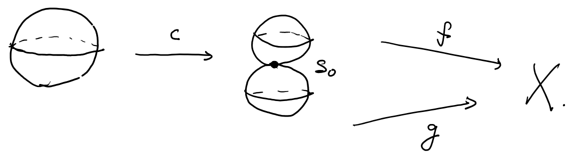
Hence, $\pi_n(X, x_0) \overset{\cong}{\longleftrightarrow} [(S^n, s_0), \ (X, x_0)]$

$\|$

$\left\{ \begin{array}{l} \text{homotopy classes of based maps} \\ (S^n, s_0) \to (X, x_0) \end{array} \right\}$

<u>Q</u>: How to see the group law on the RHS $\nearrow$?

$f+g$ = composition $S^n \xrightarrow{c} S^n \vee S^n \xrightarrow{f \vee g} X$



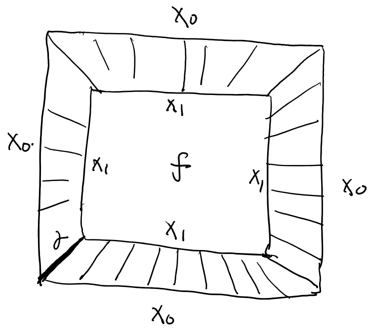
<u>prop</u>: If γ is a path in X from x_0 to x_1
then γ induces an isomorphism:

$$\gamma^*: \pi_n(X, x_1) \xrightarrow{\cong} \pi_n(X, x_0) \qquad (n \geq 1)$$
$$[f] \longmapsto [\gamma f].$$

pf: Say $\gamma: I \to X$ $\quad \gamma(0) = x_0 \quad \gamma(1) = x_1$.

Given $f: I^n \longrightarrow X$
$\qquad \partial I^n \longrightarrow x_1$

define a new map $\gamma f: I^n \longrightarrow X$ as:
$\qquad \partial I^n \longrightarrow x_0$.



$[\gamma f] \in \pi_n(X, x_0)$.

inverse: $(\gamma^*)^{-1} = (\bar{\gamma})^*$.

$\square$.

<u>Cor</u>: If X is path connected
then $\pi_n(X, x_1) \cong \pi_n(X, x_0) \quad \forall x_0, x_1$.

Rmk. The isomorphism depends on a chosen path from x_0 to x_1

Rmk: The isomorphism depends on a chosen path from x_0 to x_1.

Cor: For X path connected, we have a homomorphism:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r] & \mathrm{Aut}(\pi_n(X, x_0)) \\ \gamma \arrow[r, mapsto] & \gamma^* \end{tikzcd} \rightsquigarrow \text{prop.}$$

$\pi_1 \circlearrowright \pi_n$.

Rmk: (1) When $n=1$, $\pi_1 \circlearrowright \pi_{n=1}$ is just conjugation.

(2) We say X is <u>n-simple</u> if $\pi_1 \circlearrowright \pi_n$ trivially.

Prop: π_n is a functor. $(n \geq 2)$

$$\left\{ \begin{matrix} \text{based spaces} \\ (X, x_0) \end{matrix} \right\} \xrightarrow{\pi_n} \{ \text{ab. gp.} \}.$$

Pf:

$$\begin{tikzcd} (S^n, s) \arrow[r, "\varphi"] \arrow[rr, bend right, ""] & (X, x_0) \arrow[r, "f"] & (Y, f(x_0)) \end{tikzcd}$$

$$f\varphi \in \pi_n(Y, f(x_0))$$

□.

Rmk: (1) If $(X, x_0) \simeq (Y, y_0)$ homotopy equivalent via ~~based~~ homotopy. then $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$.

(2) (HW): A stronger property is true!

If $f: X \to Y$ is a homotopy equivalence, then $f_*: \pi_n(X, x_0) \xrightarrow{\cong} \pi_n(Y, f(x_0))$ $\forall x_0$.

Prop: For $n \geq 2$, a covering map $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces an isomorphism on π_n. $(n \geq 2)$.

Pf (p_* is onto):

Every map

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

lifts. $[f] \in \pi_n(X, x_0)$.

$(\pi_1 S^n$

Every map $S^n \xrightarrow{f} X$ lifts. $[f] \in \pi_n(X, x_0)$.

$(\pi_1 S^n = 0 \ \forall n \geq 2)$

$(p_*$ is 1-1$)$:

Every homotopy,
$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ S^n \times [0,1] \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$
lifts.

$\pi_1 = 0$

$\square$.

In particular, if X has a contractible universal cover $\tilde{X} \simeq *$.

(e.g. $X = T^n, S^1, L^\infty, \Sigma_g, \bigvee_r S^1 \ldots\ K(G,1)$).

then $\pi_n(X) = 0 \quad \forall n \geq 2$.

We say X is <u>aspherical</u>.

Prop. $\pi_n(\prod_\alpha X_\alpha) = \prod_\alpha \pi_n(X_\alpha) \quad \forall n$.

Pf: A map $f: Y \longrightarrow \prod_\alpha X_\alpha$ is the same as maps $f_\alpha: Y \to X_\alpha$. $\forall \alpha$.

Take $Y = S^n, \ S^n \times [0,1]$.

$\square$

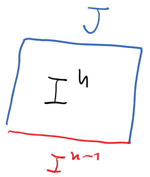
Relative homotopy groups.

$x_0 \in A \subseteq X$.

Consider $I^{n-1} \longrightarrow I^n$

$(s_1, \ldots, s_{n-1}) \longmapsto (s_1, \ldots, s_{n-1}, 0)$ $\ \leftarrow s_n = 0$.

$J^{n-1} := \overline{\partial I^n - I^{n-1}}$



Then
$n \geq 1 \rightarrow \pi_n(X, A, x_0) := \{ f: \underbrace{(I^n, \partial I^n, J^{n-1})} \longrightarrow (X, A, x_0) \} /$ homotopy of maps of triple.

(π_0 not defined).

Rmk: ① $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$

② $\pi_n(X, A, x_0)$ is a group when $n \geq 2$.

is an abelian group when $n \geq 3$.

③ $n = 1$: $I^1 = [0,1]$ $\quad I^0 = \{0\}$ $\quad J^0 = \{1\}$

$$\pi_1(X, A, x_0) = \left\{ \text{paths } \gamma \text{ in } X \text{ s.t. } \begin{array}{l} \gamma(0) = x_0 \\ \gamma(1) \in A \end{array} \right\} \Big/ \sim .$$

↑ set, not a group.

Alternatively, since

$$(I^n, \partial I^n, J^{n-1}) \Big/ J^{n-1} = (D^n, S^{n-1}, s_0)$$

$$\Rightarrow \quad \pi_n(X, A, x_0) = [\, (D^n, S^{n-1}, s_0) \,,\, (X, A, x_0) \,]$$

↑ notation: $[A, B] = \{\text{homotopy classes of maps } A \to B\}$

Rmk: ("Compression criterion")

A map $f : (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$

iff it is homotopic rel S^{n-1} to a map with image in A.

"relative to"

notation: f_t is the same map $\forall t$ when restricted to $S^{n-1} \subseteq D^n$.

Thm. There is LES:

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots$$

$$\longrightarrow \cdots \longrightarrow \pi_0(X, x_0).$$

where $i : (A, x_0) \hookrightarrow (X, x_0)$

$j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$.

Rmk: (1) Near the end, where π_0 is not a group, exactness still makes sense:

image = kernel = {elements that get mapped to the homotopy class of the constant map}

(2) $\partial : \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0)$ comes from restricting a map $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to $(S^{n-1}, s_0) \longrightarrow$

map $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to $(S^{n-1}, s_0) \to (A, x_0)$.

∂ is a group map when $n > 1$.
is a set map when $n = 1$.

pf : straightforward application of compression criterion. (skipped).

Def: A space X with based pt x_0 is n-connected
if $\pi_i(X, x_0) = 0$ $\forall i \leq n$.

TFAE : (You check).

(a) X is n-connected $\forall$ base pt.

(b) Every map $S^i \to X$ is nullhomotopic $\forall i \leq n$.

(c) . . . . extends to $D^{i+1} \to X$ $\forall i \leq n$.

Cellular Approximation Theorem.

Recall: Suppose X, Y are CW complexes.

Def: A continuous map $X \xrightarrow{f} Y$ is cellular
if $\forall k$, $f(X^k) \subseteq Y^k$. (X^k = k-skeleton in X).

Cellular approximation theorem:

Every continuous map $f: X \to Y$ of CW complexes
is homotopic to a cellular map.

Rmk. This is analogous to the simplicial approximation theorem.

pf skipped.

Same result holds for CW pairs.

Cor. $\pi_n(S^k) = 0$ $\forall n < k$.

pf: Let $S^n = e_0 \cup e_n$

Every based map $f: S^n \longrightarrow S^k$ can be homotoped to a cellular map, hence constant if $n<k$. □.

(∃ Alternative pf using differential topology. Sard's Theorem).

Cor: A CW pair (X,A) is n-connected if all cells in $X-A$ has dim $> n$.

Pf: $(D^i, \partial D^i) \xrightarrow[\text{is}]{f} (X,A)$, ~~to~~ cellular. $\quad [f] \in \pi_i(X,A)$. $\quad i \le n$.

$f(D^i) \subseteq X^i \subseteq X^n \subseteq A$. $(i \le n)$. $\quad$ compression criterion $\Longrightarrow$ $[f]=0$ □.

Cor: (X, X^n) is n-connected. $\quad (A = X^n \checkmark)$.

The inclusion $X^n \hookrightarrow X$ induces an isomorphism on π_i $\forall i<n$, a surjection on π_n.

Pf: LES:
$$\underset{\underset{0}{\shortparallel}}{\pi_{i+1}(X,X^n)} \to \pi_i(X^n) \underset{\twoheadrightarrow}{\overset{\cong}{\longrightarrow}} \pi_i(X) \longrightarrow \underset{\underset{0}{\shortparallel}}{\pi_i(X,X^n)}$$
□.

Whitehead's Theorem.

Def: A map $f: X \to Y$ is a <u>weak homotopy equivalence</u> if it induces isomorphisms on π_n $\forall n$, $\forall$ base pts.

Rmk: a homotopy equivalence (h.e.) is a weak homotopy equivalence. (w.h.e)
↑
(HW).

Q: When is the converse true?

$$\boxed{\text{Whitehead's Theorem}}$$

If X, Y are connected CW complexes, then every w.h.e. $f: X \to Y$ is a h.e.

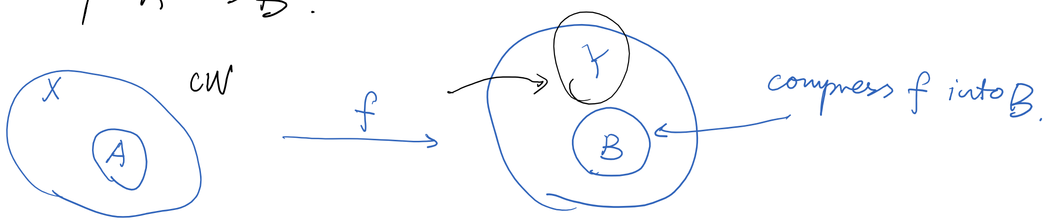
"Compression lemma":

Suppose (Y, B) is a pair of topological spaces with $B \neq \emptyset$ and (X, A) is a CW pair.

Assume that for each n s.t. $X - A$ has an n-cell,

$$(\star) \quad \pi_n(Y, B, y_0) = 0 \quad \forall\, y_0 \in B.$$

Then every map $f: (X, A) \longrightarrow (Y, B)$ is homotopic rel A to a map $X \longrightarrow B$.



pf of compression lemma: we will homotope f by ~~skeletons~~ skeletons.

Suppose $f: X^{k-1} \longrightarrow B$

We want to homotope f to $X^k \xrightarrow{f'} B$.

Consider a cell e^k of $X - A$ with characteristic map Φ:

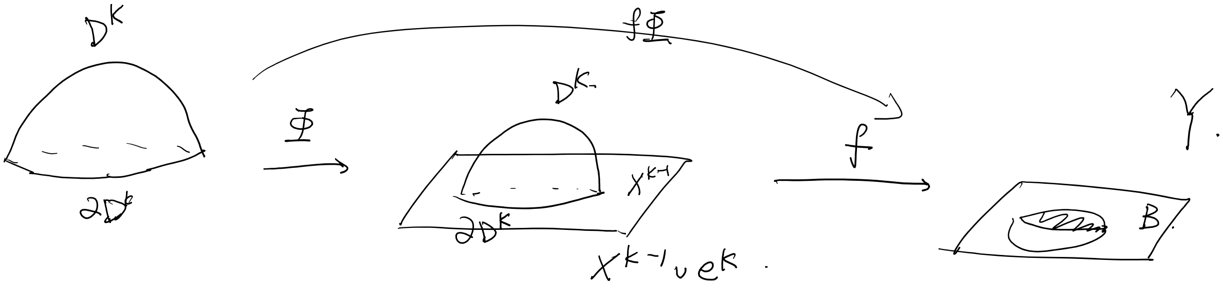
$$\begin{tikzcd}(D^k, \partial D^k) \arrow[r, "\Phi"] \arrow[rr, bend right, "f\Phi"'] & (X^k, X^{k-1}) \arrow[r, "f"] & (Y, B)\end{tikzcd}$$

$[f\Phi] \in \pi_k(Y, B) = 0$. by $(\star)$.

$\overset{\Longrightarrow}{\text{compression criterion}}$ $f\Phi$ can be homotoped rel ∂D^k to a map with image in B.

$\Longrightarrow$ f can be homotoped rel X^{k-1} to a map

$$X^{k-1} \cup e^k \overset{\text{def}}{=} X^{k-1} \coprod_{\partial D^k} D^k \longrightarrow B.$$



Finish the pf (Your job)

□.

<u>pf of Whitehead</u>: (Next week).

Lec. 23

Wednesday, May 18, 2022 1:17 PM

Whitehead's Theorem.

Def: A map $f: X \to Y$ is a weak homotopy equivalence if it induces isomorphisms on π_n $\forall n$, $\forall$ base pts.

Rmk: a homotopy equivalence (h.e.) is a weak homotopy equivalence. (w.h.e.)
↑ (HW).

Q: When is the converse true?

Whitehead's Theorem

If X, Y are connected CW complexes, then every w.h.e. $f: X \to Y$ is a h.e.

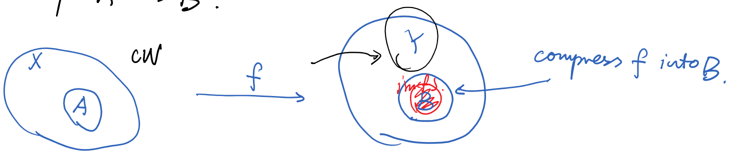
"Compression lemma":

Suppose (Y, B) is a pair of topological spaces with $B \neq \emptyset$ and (X, A) is a CW pair.

(Assume that for each n s.t. $X - A$ has an n-cell,

(☆) $\pi_n(Y, B, y_0) = 0$ $\forall y_0 \in B$.)

Then every map $f: (X, A) \longrightarrow (Y, B)$ is homotopic rel A to a map $X \to B$.



compress f into B.

Today: Proof of Whitehead's Theorem.

First consider a special case:

Claim: If $\overset{i}{\cancel{f}}: X \hookrightarrow \overset{M_f}{\cancel{Y}}$ is an inclusion of subcomplexes and is w.h.e. then Y deformation retracts onto X ($Y \searrow X$)

证: $X \hookrightarrow Y$ is a w.h.e. $\Rightarrow$ $\pi_n(Y, X) = 0$ $\forall n$.

Apply compression lemma to the identity map $(Y, X) \xrightarrow{id} (Y, X)$.

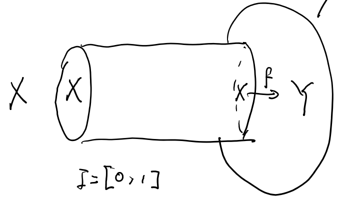
$\Rightarrow id \underset{\text{rel } X}{\simeq}$ to a map $Y \to X$ $\Leftrightarrow$ $Y \searrow X$. □.

Consider the general case: $f: X \to Y$ ~~cont.~~ cellular
$\overset{\uparrow}{\text{CW}}$ ↗

By Cellular Approximation Thm, homotope f to a cellular map.
Assume f is cellular WLOG.

Consider the "mapping cylinder":

$$M_f := (X \times I) \sqcup Y \,/\, (x,1) \sim f(x) \quad \forall x \in X.$$



<u>note</u>: ① $X = X \times \{0\} \hookrightarrow M_f$.

$Y \hookrightarrow M_f$

② $M_f \rightsquigarrow Y$. $\quad (M_f \xrightarrow[\simeq]{r} Y)$ → retraction.

③ $f : X \to Y$ factors as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, hook, "i"'] & Y \\ & M_f \arrow[u, "r"', "\simeq"] \end{tikzcd} \qquad (f \sim i).$$

It suffices to show that $M_f \rightsquigarrow X$.

$f : X \to Y$ cellular $\Rightarrow$ (M_f, X) is a CW pair.

Inclusion $i : X \hookrightarrow M_f$ is w.h.e. since:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "i"'] & Y \\ & M_f \arrow[u, "r"'] \end{tikzcd} \quad \overset{\pi_n}{\rightsquigarrow} \quad \begin{tikzcd} \pi_n X \arrow[r, "f_*", "\cong"'] \arrow[dr, "i_*"', "\cong"] & \pi_n Y \\ & \pi_n M_f \arrow[u, "r_*"', "\cong"] \end{tikzcd} \qquad \begin{array}{l} f \text{ whe.} \\ M_f \rightsquigarrow Y. \end{array}$$

claim $\Rightarrow$ Thm.

□.

<u>Warning</u>: Whitehead $\not\Rightarrow$ $\boxed{\text{CW complex } X \text{ and } Y \text{ are homotopy equivalent if they have isomorphic } \pi_n \ \forall n.}$

↳ Is false!

<u>counter-example</u>: $X = \mathbb{R}P^2$, $\quad Y = S^2 \times \mathbb{R}P^\infty$

$\pi_1(X) \cong \mathbb{Z}_2 \cong \pi_1(Y)$

$n \geq 2$: $\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2)$

$\overset{\text{lis}}{\pi_n}(S^2 \times \mathbb{R}P^\infty) \cong \pi_n(S^2 \times S^\infty) \cong \pi_n(S^2)$

X, Y have isomorphic π_n $\forall n$.

However,

$H_i(\mathbb{R}P^2) = 0 \quad \forall i > 2.$

But $H_i(S^2 \times \mathbb{R}P^\infty) = 0$ for infinitely many i's.

$H_i(\mathbb{R}P^\infty) = H_i(L_2^\infty) \neq 0$ $\forall$ many i's.

<u>CW approximation</u>

Thm: For every topological space X, there exists a CW complex Z s.t. $f: Z \to X$ is a weak homotopy equivalence. (w.h.e).

Def: Z is called a CW approximation to X.

Z ↙ CW. X ↘ top. sp.

Rmk: ① w.h.e. is the best we can hope for.

Example: quasi-circle  is not h.e. to a pt. but is w.h.e. to a pt.

② Whitehead's Thm $\Rightarrow$ CW approximation is unique up to h.e.

(suppose
$$\begin{tikzcd} Z \arrow[dr, "f"] \arrow[d, "g"'] & \\ Z' \arrow[r, "f'"'] & X \end{tikzcd}$$
CW   top. sp.

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[rr, bend right] & X \arrow[r, hook, "i"] & M_{f'} \end{tikzcd}$$

$M_{f'}$ = mapping cylinder of f'

can be homotoped to $Z \xrightarrow{g} Z'$ by compression lemma.

g is a whe. $\underset{\text{Whitehead}}{\Longrightarrow}$ g is a h.e.)

pf of CW approx.: we will build such approximation Z by skeletons.

$n=0$: $Z^0 \xrightarrow{f} X$ s.t. f is a bijection on π_0. (easy).

$\rightsquigarrow$ attach 1-cells.

Z^1

$\rightsquigarrow$

$\vdots$

$Z^n \xrightarrow{f} X$ iso on π_i $\forall i < n$. surjection on π_n.

(Induction). $\rightsquigarrow$ attach $(n+1)$-cells.

$Z^{n+1} \xrightarrow{f} X$ iso on π_i $\forall i < n+1$ surj. on π_{n+1}.

Assume, we can take

$$Z := \bigcup_{n=0}^{\infty} Z^n \xrightarrow{f} X \text{ is a w.h.e.}$$

Q: How do we do (Induction) step?

Suppose we have $\underset{\text{CW complex}}{A} \xrightarrow{f} X$

Suppose we have $\underset{\text{CW complex}}{A} \xrightarrow{f} X$

with a chosen basepoint a_γ in each component of A.

$\boxed{\text{Want}}$: For a fixed $k>0$, attach k-cells to A to form B and extend f to $B \xrightarrow{f} X$ s.t.

→ (*) : $\forall$ base pt a_γ, $f_* : \pi_i(B, a_\gamma) \longrightarrow \pi_i(X, f(a_\gamma))$

is 1-1 for $i = k-1$

onto for $i = k$

(*) will finish induction because :

Given $Z^n \xrightarrow{f} X$ iso. on π_i $\forall i<n$

surj. on π_n.

Take $A = Z^n$, $k = (n+1)$

$\overset{(*)}{\rightsquigarrow}$ $Z^{n+1} := B = A \cup (n+1)$-cells.

and $f : Z^{n+1} \longrightarrow X$

s.t. $f_* : \pi_i(Z^{n+1}, a_\gamma) \longrightarrow \pi_i(X, f(a_\gamma))$ is 1-1 $\forall i < n+1$ ↑ injective. ⇒ iso. by previous.

onto $\forall i = n+1$

Note : Attaching $(n+1)$-cells does not affect π_i $\forall i<n$. (cellular approx.).

Moreover,

$$\begin{tikzcd} \pi_n(Z^{n+1}) \arrow[r, two heads] \arrow[rr, bend right, "\text{onto.}"'] & \pi_n(Z^n) \arrow[r, two heads] & \pi_n(X) \end{tikzcd}$$

cellular approx. ↗ (two lectures ago)

↗ previous induction hypothesis.

induction ✓.

wrong →

Correction :

$$\begin{tikzcd} Z^n \arrow[r, "f"] \arrow[d, hook, "i"] & X \\ Z^{n+1} \arrow[ur, "f'"'] & \end{tikzcd}$$

$$\Rightarrow \begin{tikzcd} \pi_n(Z^n) \arrow[r, "f_*"] \arrow[d, "i_*"] & \pi_n(X) \\ \pi_n(Z^{n+1}) \arrow[ur, "f'_*"'] & \end{tikzcd}$$

⇒ if f_* onto, then f_*' onto.

$\square$

Now we will justify (*) :

Suppose we have $\underset{\text{CW complex}}{A} \xrightarrow{f} X$

with a chosen basepoint a_γ in each component of A.

$\boxed{\text{Want}}$: For a fixed $k>0$, attach k-cells to A to form B and extend $\underset{\checkmark}{\underline{f}}$ to $\underset{\checkmark}{\underline{B}} \xrightarrow{f} X$ s.t.

→ (*) : $\forall$ base pt a_γ, $f_* : \pi_i(B, a_\gamma) \longrightarrow \pi_i(X, f(a_\gamma))$

is $\underline{\text{1-1}}$ for $i = k-1$

$\underline{\text{onto}}$ for $i = k$

Step 1 : Choose maps $\varphi_\alpha : (S^{k-1}, s_0) \longrightarrow (A, a_\gamma)$ representing all nontrivial elements of

$$\operatorname{Ker}\left(f_* : \underset{[\varphi_\alpha] \nearrow [h]}{\pi_{k-1}(A, a_\gamma)} \longrightarrow \pi_{k-1}(X, f(a_\gamma)) \right) \quad \forall \text{ base pt } a_\gamma .$$

By a cellular approximation thm, assume φ_α is cellular.

(S^{k-1} has the minimal CW structure
s_0 = 0-skeleton).

Attach each k-cell e_α^k to A via φ_α $\forall \alpha$.

$$A' = \left(\coprod_\alpha D^k \right) \cup_{\varphi_\alpha} A \qquad \varphi_\alpha : \partial D^k = S^{k-1} \longrightarrow A .$$

$\forall \alpha$. $[\varphi_\alpha] \in \ker f_* \Rightarrow f\varphi_\alpha : \underset{\underset{D^k}{\cap}}{S^{k-1}} \longrightarrow A \longrightarrow X$ is nullhomotopic.

$\Rightarrow$ $f\varphi_\alpha$ extends to a map $D^k \longrightarrow X$ $\forall \alpha$.

$\Rightarrow$ f extends to

$$\begin{tikzcd} A' \arrow[r, "f'"] & X \\ A \arrow[u, phantom, "\cup"] \arrow[ur, "f"'] & \end{tikzcd}$$

Step 2: Choose $f_\beta : (S^k, s_0) \longrightarrow (X, f(a_\gamma))$ representing all nontrivial elements of $\pi_k(X, f(a_\gamma))$ $\forall$ base pt a_γ.

Attach e_β^k to A' via constant map at a_γ ←

$$A'' = \left(\coprod_\beta D^k \right) \cup_{a_\gamma} A' = \left(\bigvee_\beta S^k \right) \vee A' \xrightarrow{f} X$$

Extend f to spheres S_β^k via f_β

Step 1 + 2 $\Rightarrow$

$$\begin{tikzcd} \underline{B := A \cup_\alpha e_\alpha^k \cup_\beta e_\beta^k} \arrow[r, "f"] & X \\ A \arrow[u, phantom, "\cup"] \arrow[ur, "f"'] & \end{tikzcd}$$

$\Rightarrow B^{k-1} \subseteq A$.

Check (*) is satisfied:

surjectivity: obvious.

$$\begin{tikzcd} \pi_k(B, a_\gamma) \arrow[r, "f_*"] & \pi_k(X, f(a_\gamma)) \\ {[S_\beta^k \hookrightarrow B]} \arrow[r, mapsto] & \forall [f_\beta] \end{tikzcd}$$

✓ injectivity: Take $[h] \in \ker\left(f_* : \pi_{k-1}(B, a_\gamma) \to \pi_{k-1}(X, f(a_\gamma)) \right)$

Assume $h : S^{k-1} \to B$ cellular.

$\Rightarrow h(S^{k-1}) \subseteq B^{k-1} \underset{\nwarrow ?}{\subseteq} A. \overset{?}{\leftarrow}$ (since $B = A \cup$ k-cells.)

$\Rightarrow [h] \in \pi_{k-1}(A, a_\gamma)$.

and $[h] \in \ker\left(f_* : \pi_{k-1}(A, a_\gamma) \longrightarrow \pi_{k-1}(X, f(a_\gamma)) \right)$

By Step 1, $h \sim \varphi_\alpha$ for some α.

$$\begin{tikzcd} S^{k-1} \arrow[r, "\varphi_\alpha"] \arrow[d, hook] & B \\ D^k \arrow[ur] & \end{tikzcd}$$

$\varphi_\alpha : S^{k-1} \longrightarrow B$ extends to D^k.

$\Rightarrow$ $\overset{h\sim}{\varphi_\alpha}$ is ~~[illegible]~~ nullhomotopic as $S^{k-1} \longrightarrow B$

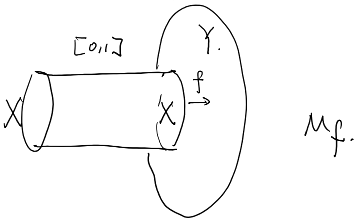
$\Rightarrow [h] = 0$.

□.

CW approximation v.s. homology

Prop. A w.h.e. $f: X \longrightarrow Y$ induces isomorphisms $f_*: H_n(X;G) \longrightarrow H_n(Y;G)$ $\forall n$ $\forall$ coefficient G.

Rmk: Same for $H^n(-;G)$ by UCT.



Pf: Consider $X \overset{i}{\hookrightarrow} M_f \rightsquigarrow Y$ (mapping cylinder), $X \xrightarrow{f} Y$.

It suffices to prove ~~[illegible]~~ prop. for $i: X \overset{i}{\hookrightarrow} M_f$. ✓

$\pi_n(M_f, X) = 0$ $\forall n$.

We know: (M_f, X) is an n-connected pair $\forall n$. (LES of π_n).

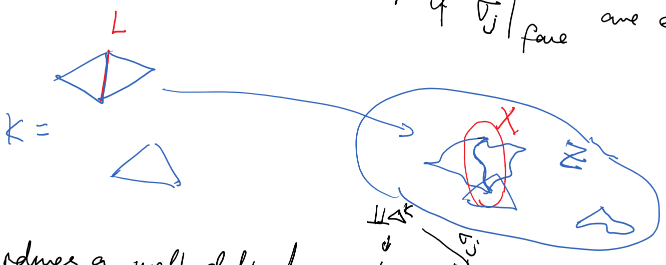
$\boxed{\text{WTS}}$: $H_i(M_f, X; G) = 0$ $\forall i$. (LES of H_i).

$H_i(X) \xrightarrow{\cong} H_i(M_f)$ ✓

claim. If (Z, X) is an n-connected pair of path connected spaces, then $H_k(Z, X; G) = 0$ $\forall k \leq n$. $\forall G$.

Pf. Take any $\overset{\in}{[\alpha]}$, $\overset{C_k(Z,X;G)}{\overset{\in}{\alpha}} = \sum_j n_j \sigma_j$, $k \leq n$, $n_j \in G$, $\sigma_j: \Delta^k \longrightarrow Z$.

Build a Δ-complex $K := \coprod_j \Delta_j^k \Big/$ identify faces of Δ_j^k if $\sigma_j|_{\text{face}}$ are equal in Z.



s.t. $\bigcup_j \sigma_j$ induces a well-defined $\sigma: K \longrightarrow Z$.

α relative cycle $\Rightarrow$ $\partial\alpha$ a $\overset{(k-1)}{\text{chain}}$ in $X \subseteq Z$.

Let $L \subseteq K$ be the subcomplex consisting of $(k-1)$-simplices that correspond to all $(k-1)$-simplices in $\partial\alpha$.

so $\sigma : (K, L) \longrightarrow (Z, X)$ continuous.

$\uparrow$ Δ-complex $\qquad$ $\uparrow$ top. spa.

$$H_k(K,L) \xrightarrow{\sigma_*} H_k(Z,X)$$

$$\exists\, [\tilde{\alpha}] \longmapsto [\alpha]$$

$\sigma_*[\tilde{\alpha}] = [\alpha]$.

compression lemma applied to $\sigma : (K, L) \longrightarrow (Z, X)$. $\dashrightarrow (X, X)$

$\uparrow$ k-dim Δ-complex. $\quad$ $\uparrow$ $(k-1)$. $\quad$ $\uparrow$ n-connected. $\quad k \leq n$.

$\Rightarrow$ σ can be homotoped rel L into X.

$\Rightarrow$ $\sigma_*[\tilde{\alpha}]$ is in the image of

$$\underset{\overset{\shortparallel}{0}}{H_k(X, X; G)} \longrightarrow H_k(Z, X; G).$$

$$\longmapsto 0 = \sigma_*[\tilde{\alpha}] = [\alpha].$$

$\Rightarrow$ $[\alpha] = 0$.

$\square$.

<u>Recall</u>: Previously, we stated Kunneth Theorem, topological version.

$$H^\cdot(X \times Y; R) \cong H^\cdot(X; R) \otimes H^\cdot(Y; R)$$

under some conditions ..... for X, Y CW complexes.

How about X, Y top. spaces?

Pick X', Y' CW approximation to X, Y, resp.

then $\underbrace{X' \times Y'}_{\text{CW complex}}$ is a CW approx. to $X \times Y$. (!)

$\uparrow$ (2 topologies with the same π_n $\forall n$).

$X' \times Y'$ is w.h.e to $X \times Y$.

So Kunneth applies to top. spaces.

Suppose $f : Y \longrightarrow Z$ is a w.h.e.

$$\Leftrightarrow \quad f_* : \underset{\pi_n Y}{\overset{\shortparallel}{[S^n, Y]}} \longrightarrow \underset{\pi_n Z}{\overset{\shortparallel}{[S^n, Z]}}. \text{ is a bijection } \forall n.$$

Fact: The same holds if we replace S^n by any complex X.

Prop. A w.h.e. $f: Y \to Z$ induces a bijection

$$f_* : [X, Y] \xrightarrow{\cong} [X, Z] \quad \text{for any CW complex } X.$$

$$(X \to Y) \longmapsto (\underbrace{X \to Y \xrightarrow{f} Z}).$$

Rmk. The functor $[X, -]$ is invariant under w.h.e.

Pf. WLOG, assume f is an inclusion $Y \xhookrightarrow{f} Z$. (replace Z by M_f)
↳ whe.

$\pi_n(Z, Y) = 0 \quad \forall n \quad \forall$ base pt.

compression lemma $\Rightarrow$ any $X \longrightarrow Z$ can be homotoped to be in Y.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, dashed] & Z \arrow[d, phantom, "\cup"] \\ & Y \end{tikzcd}$$

$\Rightarrow f_*$ is onto.

Apply same argument to a homotopy $(X \times I, X \times \partial I) \longrightarrow (Z, Y)$.

$\Rightarrow f_*$ is 1-1. ◻

Calculating π_n (elementary methods)

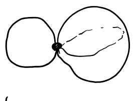
↳ not using spectral seq.
↳ next semester.

Excision for π_n.

$n = 1$. van Kampen Thm.

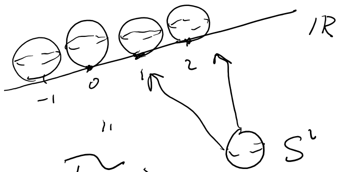
Rmk: Excision fails for π_n $n \geq 2$.

Example: $\pi_2(S^1 \vee S^2)$



$\pi_2 S^1 = 0 \quad \pi_2 S^2 = \mathbb{Z}$.

$\pi_2(*) = 0$.



$$\pi_2(\widetilde{S^1 \vee S^2}) \cong \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}.$$

But $\pi_2(S^1 \vee S^2)$ is not finitely generated

Thm. ("excision" for π_n).

Let X be a CW complex s.t. $X = A \cup B$

A, B subcomplexes with $A \cap B = C$ connected.

If (A, C) is m-connected

(B, C) is n-conneted $\quad m, n \geq 0$.

then the inclusion $(A, C) \hookrightarrow (X, B)$ induces

an iso on $\pi_i \quad \forall\, i < m+n$

and a surjection on $\pi_i \quad i = m+n$.

pf skipped.

Cor. (Freudenthal suspension thm).

If X is an $(n-1)$-connected CW cmplex then the suspension map

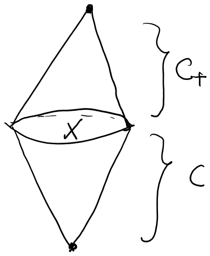
$$\pi_i(X) \longrightarrow \pi_{i+1}(SX)$$

↳ suspension of X

is an iso. for $i < 2n-1$.

and a surjection for $i = 2n-1$.

Recall: $SX =$



C_+X cone on $X \simeq *$.

C_-X. cone on X

$$SX = C_+X \cup_X C_-X, \quad C_+X, C_-X \simeq *.$$

$$\pi_i(X) \underset{\substack{\downarrow \\ \text{LES } (C_+X, X).}}{\cong} \pi_{i+1}(C_+X, X) \underset{\substack{\downarrow \\ \text{Excision Thm} \quad (C_+X, X) \hookrightarrow (SX, C_-X).}}{\xrightarrow{\cong}} \pi_{i+1}(SX, C_-X) \overset{\nearrow \text{ LES } (SX, C_-X).}{\cong} \pi_{i+1}(SX)$$

□.

Rmk. By Cor. X is $(n-1)$-connected $\boxed{n \geq 2}$.

$\Rightarrow SX$ is n-conneted.

$\Rightarrow S^2X$ is $(n+1)$-connected

$\vdots$

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \cdots \xrightarrow{\cong} \pi_{i+N}(S^NX) \xrightarrow{\cong}$$

↗ $\pi_i^S(X)$ "stable homotopy group of X"

$\downarrow$

eventually becomes iso.

$\underset{\sim\sim}{i < 2n-1}$ eventually satisfied if we increase i and n at the same speed.

For spheres, $X = S^0$. $\boxed{S(S^n) = S^{n+1}}$.

$$\pi_i^s(S^0) \cong \pi_{i+n}(S^n) \quad \forall n > i+1.$$

"stable homotopy group of spheres".

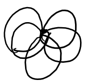
<u>HW</u>: $\pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$. $\forall n$. $\Rightarrow \pi_0^s(S^0) \cong \mathbb{Z}$

$f \longmapsto \deg(f)$

<u>Thm</u>. (Serre). $\pi_i^s(S^0)$ is finite $\forall i > 0$ (next semester).

<u>Example</u>: $n \geq 2$. $\pi_n(\overset{\nearrow \text{wedge sum}}{\bigvee_\alpha} S_\alpha^n) \cong \bigoplus_\alpha \underset{\underset{\mathbb{Z}}{\shortparallel}}{\pi_n(S_\alpha^n)}$ for any number of wedges.

pf for finitely many wedge:



$$\bigvee_\alpha S_\alpha^n = n\text{-skeleton of } \prod_\alpha S_\alpha^n \longleftarrow \text{product CW complex.}$$

with only cells of dim $= kn$, $k \in \mathbb{N}$.

$\Rightarrow (\prod_\alpha S_\alpha^n, \bigvee_\alpha S_\alpha^n)$ is $(2n-1)$ connected.

$$\overset{\text{LES}}{\underset{(n \geq 2)}{\Rightarrow}} \pi_n(\bigvee_\alpha S_\alpha^n) \cong \pi_n(\prod_\alpha S_\alpha^n) \overset{\cong \prod_\alpha \mathbb{Z} \ \shortparallel}{\cong} \bigoplus_\alpha \mathbb{Z}.$$

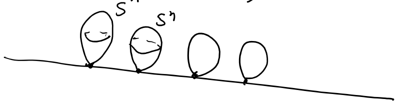
$\uparrow$ finitely many α.

Extra work $\Rightarrow$ infinite wedge.

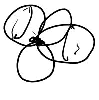
<u>Ex</u>: $\pi_n(S^1 \vee S^n) \overset{\pi_1 \curvearrowright \ (n\geq 2)}{\cong} \pi_n(\bigvee_{\mathbb{Z}} S^n) \overset{\text{previous}}{\cong} \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$.

$\shortparallel$

$\pi_n(\widetilde{S^1 \vee S^n})$



$\simeq$



$= \bigvee_{\mathbb{Z}} S^n$

<u>Rmk</u>: Finite CW complexes might have infinitely generated π_n. (in contrast to H_n).

Thm (Serre): If $\pi_1 \circlearrowright \pi_n$ trivially $\forall n (\leq N)$

then π_n is fin. gen. $\forall n (\leq N)$

iff H_n is fin. gen. $\forall n (\leq N)$

(Next semester).

Rmk (vague)

	wedge sum	product ⇝ fiber bundle		sphere	aspherical spaces. Eg. graphs.
π_n	☹	☺	☺ (later). next week	☹	☺
H_n.	☺	☹ Kunneth	☹ (next semester spectral seq.)	☺	☹ group homology.

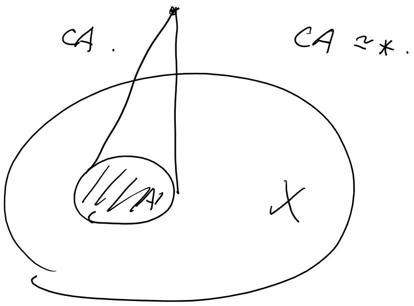
prop. If a CW pair (X,A) is r-connected

and A is s-connected $r,s \geq 0$.

then $X \longrightarrow X/A$ induces:

an iso. on π_i $\forall i \leq r+s$.

a surjection on π_i $i = r+s+1$.



Pf. $X \cup CA$.

$$X \cup CA \xrightarrow{\sim} \frac{X \cup CA}{CA} = X/A$$

↓ is a h.e.

$$\begin{tikzcd}
\pi_i(X,A) \arrow[r, "\overset{\cong\ \forall i \leq r+s.}{\twoheadrightarrow\ i=r+s+1.}"] & \pi_i(X \cup CA,\ \overset{\overset{*}{\text{is}}}{CA}) \arrow[r, "\cong"] & \pi_i\left(\frac{X \cup CA}{CA}\right) \cong \pi_i(X/A). \\
 & \pi_i(X \cup CA) \arrow[u, "\cong", "\text{LES.}"'] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

↑ Apply excision.

↑

need: (CA, A) is $(s+1)$-connected.

⇑

A is s-connected, $CA \simeq *$, LES of π_i $\square$.

Lec. 25

Wednesday, May 25, 2022 1:27 PM

Today:
- Eilenberg-Maclane spaces
- Hurewicz theorem.

Eilenberg - MacLane spaces

Def: A space X with only one nontrivial $\pi_n(X) \cong G$ is called an Eilenberg-MacLane space $K(G,n)$.

↘ abelian if $n \geq 2$.

$$\pi_i(X) = \begin{cases} G & i=n \\ 0 & i \neq n. \end{cases}$$

Recall: $K(G,1)$ = aspherical spaces (with universal cover contractible).

Rmk: There are not many naturally occuring $K(G,n)$'s for $n \geq 2$.
except : $K(\mathbb{Z},2) = \mathbb{C}P^\infty$ (later).

✓ Existence of $K(G,n)$ ($n \geq 2$, G = ab. gp).

construction in 2 steps:

$\boxed{\text{Step 1}}$: Build a CW complex $X = X^{n+1}$ that is

- ✓ $(n+1)$-dimensional
- ✓ $(n-1)$-connected
- ✓ $\pi_n(X) \cong G$.

In general, let $X^n := \bigvee_\alpha S^n_\alpha$
and attach cells e_β^{n+1} via based maps $\varphi_\beta : S^n \longrightarrow X^n$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\cap$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad D^{n+1}.$
to get $X^{n+1} =: X$.

claim: $(n \geq 2)$: $\pi_n(X^{n+1}) = \dfrac{\pi_n(X^n)}{\langle [\varphi_\beta] \rangle_\beta}$ $\quad \pi_n(X^n) \longrightarrow \cong \bigoplus_\alpha \mathbb{Z}$

[You can seen this when $n=1$]

Pf: LES of pairs:

$$\begin{tikzcd} \pi_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial"] \arrow[d, "\cong"] & \pi_n(X^n) \arrow[r] \arrow[d, equal] & \pi_n(X^{n+1}) \arrow[r] & 0 \\ \pi_{n+1}(X^{n+1}/X^n) & \bigoplus_\alpha \mathbb{Z}. & & \end{tikzcd}$$

⇝ prop last time
excision.

$1_\beta \xrightarrow{\partial} [\varphi_\beta]$.

$\rightsquigarrow$ prop last time IIS $I_\beta \mapsto [\varphi_\beta]$. $\parallel$

excision. $\pi_{n+1}(X^{n+1}/X^n)$ $\bigoplus_\alpha \mathbb{Z}$.

$$\parallel$$

$$\pi_{n+1}\left(\bigvee_\beta S_\beta^{n+1}\right)$$

$$\parallel$$

$$\bigoplus_\beta \mathbb{Z}.$$

$\square$.

$\langle g_\alpha \mid r_\beta = 0 \rangle$.

$\parallel$

Given G ab. gp, we have

$$\bigoplus_\beta \mathbb{Z} \longrightarrow \bigoplus_\alpha \mathbb{Z} \longrightarrow G \longrightarrow 0$$

free ab. free ab.

claim

$\Rightarrow$ $\pi_n(X^{n+1}) \cong G$.

Moreover, $\pi_i(X^{n+1}) = 0$ $\forall i \leq n-1$.

[consider $f: S^i \to X^{n+1}$, $f \sim$ a cellular map.

$f(S^i) \subseteq X^i$ = a point ($i \leq n-1$).

$[f] = 0$. $\uparrow$ i-skeleton of X^{n+1}].

In summary, $\pi_i(X^{n+1}) = \begin{cases} 0 & i < n. \\ G & \underline{\underline{i = n}} \\ \textcircled{?} & i > n \end{cases}$

Goal: kill $\pi_{>n}$.

Step 2: Attach higher dim cells to X^{n+1} to kill higher homotopy groups. (cellular).

To kill $\pi_{n+1}(X^{n+1})$, we take $\varphi_\gamma : S^{n+1} \longrightarrow X^{n+1}$ representing all nontrivial elements in $\pi_{n+1}(X^{n+1})$.

Attach e_γ^{n+2} to X^{n+1} via φ_γ and obtain X^{n+2}.

We have: $\pi_i(X^{n+2}) \cong \pi_i(X^{n+1})$ $i \leq n$.

$\nwarrow$ cellular approx.

$i = n+1$. $\pi_{n+1}(X^{n+2}) = 0$. ✓

$$\left[\begin{array}{l} \pi_{n+1}(X^{n+1}) \twoheadrightarrow \pi_{n+1}(X^{n+2}) \to 0. \\ \quad [\varphi_\gamma] \longmapsto \left[\begin{tikzcd} S^{n+1} \arrow[r, "\varphi_\gamma"] \arrow[rr, bend right] \arrow[d, phantom, "\cap"] & X^{n+1} \arrow[r, hook] & X^{n+2} \\ D^{n+2} \arrow[urr, dashed] & & \exists.\ e_\gamma^{n+2}. \end{tikzcd}\right] = 0 \end{array}\right]$$

Inductively, attach cells and obtain X^m ...

$$\begin{tikzcd} \cup\!\!\shortmid \arrow[rr, bend right, dashed] & & \\ D^{n+2} \arrow[rru, dashed] & \exists.\ e_\gamma^{n+2}. & \end{tikzcd}$$

Inductively, attach cells and obtain X^m $m = n+2, n+3, n+4 \dots$

$$X = \bigcup_{m \geq 0}^{\infty} X^m .$$

$$\pi_i(X) = \begin{cases} 0 & i < n \\ G & i = n \\ 0 & i > n . \end{cases} \Rightarrow X \text{ is a } K(G,n).$$

Uniqueness of $K(G,n)$:

The homotopy type of a CW complex $K(G,n)$ is uniquely determined by G and n. ⇝ unique up to homotopy.

Pf: Take any Y, a $K(G,n)$.

Let X be before. Goal: Build a h.e. $f: X \xrightarrow{\sim} Y$.

Note: we have an isomorphism of groups. ↙

$$\begin{tikzcd} \pi_n(X) \arrow[r, "\Psi"] \arrow[d, phantom, "\shortparallel"] & \pi_n(Y). \\ \pi_n(X^{n+1}) & \\ \pi_n(X^n) \arrow[u, two heads] \arrow[ruu, "\Psi"] & \end{tikzcd}$$

$[\varphi_\beta] = 0$ $\quad$ $\ni [\varphi_\beta] \neq 0$

$[i_\alpha : S^n_\alpha \hookrightarrow X^n]$

$X^n = \bigvee_\alpha S^n_\alpha$

$\Psi[i_\alpha].: S^n \longrightarrow Y.$

pick a representative. (up to homotopy).

Plan: Extend $\Psi[i_\alpha] : S^n \longrightarrow Y$ to $X^{n+1} \xrightarrow{f} Y.$

s.t. $f_*[i_\alpha] = \Psi[i_\alpha].$

$$\begin{tikzcd} X^{n+1} \arrow[r, "f"] & Y. \\ \bigvee_\alpha S^n_\alpha \arrow[u, phantom, "\shortparallel"] \arrow[ru, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} \exists\ X^n = \bigvee_\alpha S^n_\alpha \arrow[r, "f"] \arrow[d, phantom, "\cap"] & Y. \\ X^{n+1} \arrow[ru, dashed, "\exists ?"'] & \end{tikzcd}$$

↗ with char. map $\varphi_\beta : S^n \to X^n$.

To extend f over a cell e_β^{n+1}, it suffices to check that

$$\begin{tikzcd} S^{\overset{n+1}{\textcircled{n}}} \arrow[r, "\varphi_\beta"] \arrow[d, phantom, "\cap"] & X^n \arrow[r, "f"] & Y. \\ D^{n+1} \arrow[rru, dashed] & & \end{tikzcd}$$

is nullhomotopic.

or $0 = [f \circ \varphi_\beta] = f_*[\varphi_\beta] = \Psi[\varphi_\alpha]$ ✓

or $\overset{\cup}{0} = [f\circ\varphi_\beta] = f_*\underset{\pi_n(X^n)}{\underset{\cap}{[\varphi_\beta]}} = \Psi[\varphi_\beta]$ ✓

$\exists$ extension $f: X^{n+1} \longrightarrow Y$. inducing iso. Ψ on π_n.

We can extend $f: X^{n+2} \longrightarrow Y$ since $\pi_{n+1}(Y)=0 \Leftarrow Y$ is $K(G,n)$.

Inductively, extend f to X^k $\forall k \geq n$.

so $f: X = \bigcup_k X^k \longrightarrow Y$. cont.

f is a w.h.e. $(f_* = \Psi$ iso. on $\pi_n)$.

$\overset{\text{Whitehead}}{\Longrightarrow}$ f is a h.e. □.

Rmk. We can apply the same argument without assuming Ψ is an iso.

Cor. A group homomorphism $\Psi: G \longrightarrow G'$ is induced by a continuous map $f: K(G,n) \rightarrow K(G',n)$ (so. $f_* = \Psi$).

Hurewicz Theorem. (relating π_n and H_n).

Thm (Hurewicz). If a topological space X is $(n-1)$-connected $(n\geq 2)$.
then $\tilde{H}_i(X)=0$ $\forall i<n$.
and $\underset{\overset{\uparrow}{\text{first nontrivial homotopy group.}}}{\pi_n(X)} \cong H_n(X)$.

$\Rightarrow$ $H_i(K(G,n)) = \begin{cases} 0 & i<n \\ G & i=n \\ ? & i>n \end{cases}$

$\nearrow$ spectral seq.

Example. S^n is $(n-1)$-connected. (cellular approx.)

$\overset{\text{Thm}}{\Longrightarrow}$ $\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}$.

But S^n might have nontrivial $\pi_{>n}$. (e.g. $\overset{H_3(S^2)=0.}{\overset{\neq}{\pi_3(S^2)}} \cong \mathbb{Z}$).

pf of Hurewicz: By CW approx., we assume X is a CW complex.

X $(n-1)$-connected $\underset{?}{\Rightarrow}$ X is h.e. to a CW complex X with X^{n-1} = a point

(HW reading.
prop. 4.15).

(HW reading.
prop. 4.15).

We can just take $\underset{\sim\sim}{X = X^{n+1}}$ with affecting π_n and H_n.

So $X = X^{n+1} = \left(\bigvee_\alpha S_\alpha^n\right) \cup_\beta e_\beta^{n+1}$

$\uparrow$ n-cells $\qquad \uparrow$ (n+1)-cells

As before.

$$\begin{tikzcd}
 & & \bigvee_\alpha S_\alpha^n & & \\
 & \pi_{n+1}(X^{n+1},X^n) \arrow[r, "\partial"] \arrow[d, "\cong"] & \pi_n(X^n) \arrow[r] \arrow[u] \arrow[dd, equal] & \pi_n(X^{n+1}) \arrow[r] \arrow[ddd, "\cong"] & 0 \\
\text{excision} \arrow[ur] & \pi_{n+1}(X^{n+1}/X^n) \arrow[d, "\cong"] \arrow[dl] & & & \\
\bigvee_\beta S_\beta^{n+1} & \bigoplus_\beta \mathbb{Z} \arrow[r] \arrow[d, equal] & \bigoplus_\alpha \mathbb{Z} \arrow[d, equal] & & X = X^{n+1} \\
 & H_{n+1}(X^{n+1},X^n) \arrow[r, "\partial"] & H_n(X^n,X^{n-1}) \arrow[r] & H_n^{CW}(X) \arrow[r] \arrow[ur] & 0 \\
 & & \text{cellular chain complexes} \arrow[u] \arrow[ul, bend left] & &
\end{tikzcd}$$

$\square$.

$\underline{\text{Cor}}$. (Relative Hurewicz).

If (X,A) is $(n-1)$-connected $(n \geq 2)$, A is 1-connected,

then $H_i(X,A) = 0 \quad \forall i < n$.

$\pi_n(X,A) \cong H_n(X,A)$.

Pf. Assume (X,A) is a CW pair.

$\pi_i(X,A) \cong \pi_i(X/A) \quad \forall i \leq n$. by excision (last lecture).

$H_i(X,A) \cong \tilde{H}_i(X/A) \; \forall i$ by excision for homology.

reduced to the absolute case. $\square$.

$\underline{\text{Cor}}$. A map $f: X \to Y$ between $\underline{\text{1-connected}}$ CW complexes

is a h.e. if it induces iso. on $H_n \quad \forall n$.

Pf. Assume f is an inclusion $X \hookrightarrow Y$. (by replacing Y by M_f).

π_n. (Whitehead).

Relative Hurewicz $\Rightarrow$ the first nontrivial $\pi_n(Y,X)$ is isomorphic to $H_n(Y,X)$

Relative Hurewicz $\Rightarrow$ the first nontrivial $\pi_n(Y,X)$ is isomorphic to $H_n(Y,X)$

$$\overset{\shortparallel}{0} \leftarrow \text{by LES of } H.$$

$\Rightarrow \pi_n(Y,X) = 0 \quad \forall n.$

$\Rightarrow f$ is a w.h.e. $\overset{\text{whitehead}}{\Longrightarrow}$ f is h.e. $\square$.

Hurewicz map.

$\forall n.\ \forall X.$

$$\begin{tikzcd} \pi_n(X) \arrow[r, "h"] & H_n(X). \\ {[f: S^n \to X]} \arrow[r, mapsto] & f_*[S^n] \end{tikzcd}$$

$\overset{\text{choose}}{[S^n]} \in H_n(S^n).$

Relative:

$$\begin{tikzcd} \pi_n(X,A) \arrow[r, "h"] & H_n(X,A). \\ {[f:(D^n,\partial D^n) \to (X,A)]} \arrow[r, mapsto] & f_*[D^n] \end{tikzcd}$$

, $\overset{\text{choose}}{[D^n]}$ is a generator of $H_n(D^n,\partial D^n) \cong \mathbb{Z}$.

$\uparrow$ LES of H.

Rmk: ① $n=1$. h is the abelianization map $\pi_1 \longrightarrow H_1 = \pi_1^{ab}$. (HW).

② h is homomorphism

③ h induces the iso in the Hurewicz thm.

Fibrations.

Overview: $\exists$ two notions of "SES of spaces" $\leftarrow$ not rigorously correct.
that are "dual" to each other.

cofibration: $A \hookrightarrow X \xrightarrow{q} X/A$

fibration: $F \to E \xrightarrow{p} B$

	cofibration	fibration
H_n	☆ ☺ (excision) for H_n.	☹ (spectral seq. ← next sem.
π_n	☹ (excision for π_n)	☆ ☺ (today).

...	for H_n.	seq. next sem.
π_n	☹ (excision for π_n)	☆ ☺ (today).

Def: A map $p: E \to B$ has the homotopy lifting property for X (HLP for X).
if given a homotopy $g_t : X \to B$, $t \in I$
and a map $\tilde{g}_0 : X \to E$ lifting $g_0 : X \to B$.
then there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t $\forall t \in I$.

$$\begin{tikzcd}
X \times \{0\} \arrow[r, dashed, "\tilde{g}_0"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "g_t"'] \arrow[ur, dashed, "\exists \tilde{g}_t"] & B
\end{tikzcd}$$

A fibration is a map $p: E \to B$ satisfying the HLP for all spaces X.
"base"

"Dually", a map $i: A \to X$ has the homotopy extension property for Y (HEP for Y)
if for any map $f_0 : A \to Y$ that extends to $\tilde{f}_0 : X \to Y$
we can extend a homotopy $f_t : A \to Y$ to
a homotopy $\tilde{f}_t : X \to Y$.

$$\begin{tikzcd}
Y & X \arrow[l, "\tilde{f}_0"'] \arrow[dl, phantom, "\tilde{f}_t"] \\
\{\varphi : I \to Y\} \arrow[u, "p_0"] & A \arrow[l, dash] \arrow[u, "i"']
\end{tikzcd}$$

$i: A \to X$ is a cofibration if it has HEP for all spaces Y.

Example: $A \subseteq X$ closed $i: A \hookrightarrow X$ is a cofibration.

We saw that if (A, X) is a good pair.

$$A \to X \twoheadrightarrow A/X \overset{\text{excision}}{\rightsquigarrow} \text{LES of } \tilde{H}_n .$$

Today: $F \to E \xrightarrow{p} B$ fibration ⇝ LES of π_n.

Thm: Suppose $p: E \to B$ has the HLP for disks D^k $\forall k \geq 0$.
Choose basepoints $b_0 \in B$ and $x_0 \in \underline{F = p^{-1}(b_0)}$
Then $p_*: \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0)$ is an isomorphism $\forall n \geq 1$.
Hence, if B is path-connected, then $\exists$ LES of π_n.

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \overset{\substack{\pi_n(E,F,x_0). \\ \cong}}{\pi_n(B, b_0)} \to \cdots \to \pi_0(E, x_0) \to 0.$$

Rmk/Def: HLP for all D^k $\overset{\text{exercise.}}{\iff}$ HLP for all pairs $(\underline{D^k, \partial D^k})$.
$\iff$ HLP for all CW complexes.

$p: E \to B$ is a Serre fibration if it has $\uparrow$

Example (later): fiber bundles are Serre fibrations.

Pf 1 (p_* is onto): $\forall [f] \in \pi_n(B, b_0)$

$$f: (I^n, \partial I^n) \longrightarrow (B, b_0).$$ map of pairs.

Consider
$$\begin{array}{ccl} I^n & = & [0,1] \times I^{n-1} \\ \cup| & & \quad\uparrow \\ I^{n-1} & = & \{0\} \times I^{n-1} \subseteq \partial I^n. \end{array} \quad \Rightarrow f(I^{n-1}) = \{b_0\}.$$

$$\begin{tikzcd} I^{n-1} \arrow[r] \arrow[d, hook] & E \ni x_0 \arrow[d, "p"] \\ I^n = [0,1] \times I^{n-1} \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & B \ni b_0 \end{tikzcd}$$

p has HLP for I^{n-1}.

$f(\partial I^n) = b_0 \Rightarrow \tilde f(\partial I^n) \subseteq p^{-1}(b_0) = F.$

$\Rightarrow \tilde f : (I^n, \partial I^n, J^{n-1}) \longrightarrow (E, F, x_0).$

$$\begin{tikzcd} \Rightarrow \lbrack\tilde f\rbrack \in \pi_n(E,F,x_0) \arrow[r, "p_*"] \arrow[dr, bend right] & \pi_n(B,b_0) \\ & \lbrack f\rbrack . \end{tikzcd}$$

since $p\tilde f = f \Rightarrow p_*([\tilde f]) = [f].$

(p_* is 1-1): Similar argument.

(say $p_*[f_0] = p_*[f_1] = [f]$, find representative $\tilde f_0, \tilde f_1$ s.t. $p\tilde f_0 \sim p\tilde f_1$ in B $\xrightarrow[\text{argument}]{\text{same}}$ lift homotopy to $\tilde f_0 \sim \tilde f_1$ in E.) □.

Q: What are some examples of Serre fibrations?

A: Fiber bundles.

Def: A fiber bundle with fiber F is a map $p: E \to B$ s.t. each point in B has a neighborhood U and a homeo h s.t.

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "h", "\cong"'] \arrow[d, "p"'] & U \times F \arrow[d] & (u,f) \arrow[d, mapsto] \\ U \arrow[r, phantom, "="] & U & u \end{tikzcd}$$

Rmk:
- Trivial bundle $E = B \times F \xrightarrow{p} B$.
- h is a "local trivialization".
- $\forall b_1, b_0 \in U$, $p^{-1}(b_0) \cong p^{-1}(b_1) \cong F$ "fiber".
- If base B is path connected, then $p^{-1}(b_0) \cong p^{-1}(b_1) \cong F$ $\forall b_0, b_1 \in B$.

then $p^{-1}(b_0) \cong p^{-1}(b_1) \cong F \ \forall b_0, b_1 \in B$.

prop. A fiber bundle $p: E \to B$ is a Serre fibration i.e. has HLP for all disks.

pf (later).

Ex 1 (covering spaces).

$F =$ discrete space.

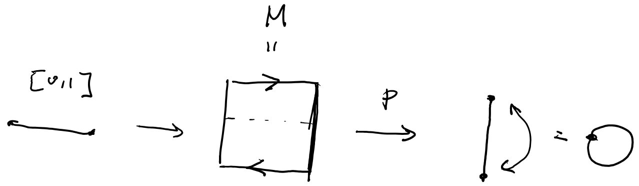
A fiber bundle $F \to E \xrightarrow{p} B$ is a covering space.

Thm + Prop. $\Rightarrow$. SES:

$$0 \to \pi_1 E \to \pi_1 B \to \pi_0 F \to \pi_0 E = 0$$

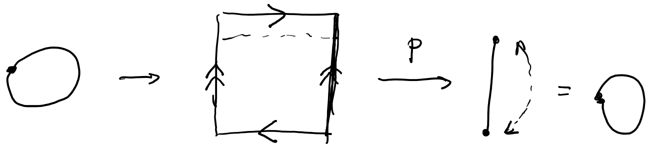
$\Rightarrow \pi_0(F) \cong F \cong \pi_1 B / \pi_1 E$. $\Leftarrow$ covering space theory. E path connected.

Ex 2: Möbius band M (nontrivial bundle over S^1.)



$$\begin{tikzcd}[0,1] \arrow[r] & M \arrow[r, "p"] \arrow[d, phantom, "\ncong"] & S^1 \\ & S^1 \times [0,1]. & \end{tikzcd}$$

Klein bottle K



$$\begin{tikzcd}S^1 \arrow[r] & K \arrow[r, "p"] \arrow[d, phantom, "\ncong"] & S^1 \\ & S^1 \times S^1 = T^2 & \end{tikzcd}$$

Ex 3: Projective spaces:

$$\begin{tikzcd}\{\pm 1\} = S^0 \arrow[r] \arrow[d, phantom, "\cap"] & S^n \arrow[r, "p"] \arrow[d, phantom, "\cap"] & \mathbb{RP}^n = S^n/S^0 \\ \mathbb{R}^{\times} \arrow[r, phantom, "\curvearrowright"] & \mathbb{R}^{n+1} \setminus 0 & \end{tikzcd}$$

$$\begin{tikzcd}S^1 \arrow[r] \arrow[d, phantom, "\cap"] & S^{2n+1} \arrow[r, "p"] \arrow[d, phantom, "\cap"] & \mathbb{CP}^n = S^{2n+1}/S^1. \\ {} & {} & \end{tikzcd}$$

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, phantom, "\cap"] & S^{2n+1} \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{C}P^n = S^{2n+1}/S^1. \\ \mathbb{C}^\times \arrow[r, phantom, "\curvearrowright"] & \mathbb{C}^{n+1}\setminus 0 & \end{tikzcd}$$

$\boxed{n \to \infty}$ fiber bundle: $S^1 \longrightarrow S^\infty \xrightarrow{p} \mathbb{C}P^\infty$

$$\begin{tikzcd} S^\infty \arrow[d, phantom, "\cong" rotate=90] \\ * \end{tikzcd}$$

Thm: LES of π_n $\Rightarrow$ $\pi_n(\mathbb{C}P^\infty) \cong \pi_{n-1}(S^1) = \begin{cases} \mathbb{Z} & n=2 \\ 0 & n \neq 2 \end{cases}$

$\Rightarrow$ $\mathbb{C}P^\infty$ is a $K(\mathbb{Z},2)$!

$\boxed{n=1}$.

$S^1 \longrightarrow S^3 \xrightarrow{p} \mathbb{C}P^1 \cong \mathbb{C}\cup\{\infty\} \cong S^2$. ⟵ "Hopf bundle"

LES of π_n $\Rightarrow$

$$\begin{tikzcd} \pi_3(S^1) \arrow[r] \arrow[d, phantom, "\parallel"] & \pi_3(S^3) \arrow[r, "p_*", "\cong"'] \arrow[d, phantom, "\cong" rotate=90] & \pi_3(S^2) \arrow[r] & \pi_2(S^1) \arrow[d, phantom, "\parallel"] \\ 0 & \mathbb{Z} & & 0 \end{tikzcd}$$

$\Rightarrow$ $\pi_3(S^2) \cong \mathbb{Z} = \langle p \rangle$.

$\uparrow$

a generator is given by $$\begin{tikzcd} S^3 \arrow[r, "id"] \arrow[rr, bend right, "p"'] & S^3 \arrow[r, "p"] & S^2 \end{tikzcd}$$ Hopf bundle.

Ex4. Replace $\mathbb{C}$ by $\mathbb{H}$ "quaternions" $\mathbb{H} = \{a+bi+cj+dk\} \cong \mathbb{R}^4$

$$\begin{tikzcd} \mathbb{H}\setminus 0 \arrow[r] & \mathbb{H}^{n+1}\setminus 0 \arrow[r] & \mathbb{H}P^n \\ S^3 \arrow[r] \arrow[u, phantom, "\cup" rotate=90] & S^{4n+3} \arrow[r] \arrow[u, phantom, "\cup" rotate=90] & \mathbb{H}P^n \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

$n=1$: $\mathbb{H}P^1 = \mathbb{H}\cup\{\infty\} \cong S^4$

⇝ Hopf bundle for $\mathbb{H}$: $S^3 \longrightarrow S^7 \longrightarrow S^4$.

$(n=\infty)$: But S^3 is not a $K(\mathbb{Z},3)$. , so $\mathbb{H}P^\infty$ is not $K(\mathbb{Z},n)$

Similarly, $\mathbb{O}$ = "octonions" $\cong \mathbb{R}^8$

($\mathbb{O}$ is not associative $\Rightarrow \nexists\ \mathbb{O}P^n \quad n \geq 2$.)

$\overset{n=1 \checkmark}{\rightsquigarrow} \quad S^7 \to S^{15} \to \mathbb{O}P^1 \cong S^8 = \mathbb{O} \cup \{\infty\}$.

<u>EX5</u> (Bott periodicity).

$$O(n-1) \subseteq O(n) \overset{\circlearrowright S^{n-1} \atop \cap}{\underset{\circlearrowright \mathbb{R}^n \setminus 0}{}}$$

$$A \longmapsto \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right].$$

$$\boxed{O(n)/O(n-1) \cong S^{n-1}}$$

$O(n-1) \to O(n) \xrightarrow{p} S^{n-1}$ bundle.

LES of π_n + S^{n-1} is $(n-2)$-connected.

$\Rightarrow$ $O(n-1) \hookrightarrow O(n)$ induces an iso. on π_i $\underline{\forall\, i < n-2.}$

$\Rightarrow$ $\pi_i\, O(n)$ independent of n if $n \to \infty$.

$\uparrow$ stabilizes as $n \to \infty$.

let $\pi_i O := \lim\limits_{n\to\infty} \pi_i O(n)$. stable homotopy gp of $O(n)$.

<u>Bott periodicity thm</u>: $\pi_i O$ is periodic in i with period 8.

(Next semester).

i	0	1	2	3	4	5	6	7
$\pi_i O$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

similar results hold for $\underset{\circlearrowright \mathbb{C}^n}{U(n)}$ and $\underset{\circlearrowright \mathbb{H}^n}{Sp(n)}$

<u>EX5</u>: (Configuration spaces).

<u>Recall</u>: $P\mathrm{Conf}^n \mathbb{C} = \{ (z_1, \dots, z_n) \mid z_i \in \mathbb{C},\ z_i \neq z_j \quad i \neq j \}$.

$\pi_1 (P\mathrm{Conf}^n \mathbb{C}) = P$

$\mid z_i \in \mathbb{C},\ z_i \neq z_j \quad i \neq j \}.$

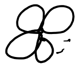
$\pi_1(P\mathrm{Conf}^n \mathbb{C}) = P_n$ "pure braid group"

We stated that $P\mathrm{Conf}^n \mathbb{C}$ is $K(P_n, 1)$.

(so $P\mathrm{Conf}^n \mathbb{C}$ is fini dim. $\Rightarrow$ P_n is torsion free).

pf. Induct on n.

$$\begin{tikzcd} P\mathrm{Conf}^{n+1}\mathbb{C} \arrow[r, two heads, "p"] & P\mathrm{Conf}^n \mathbb{C} \\ (z_1, \dots, z_{n+1}) \arrow[r, mapsto] & (z_1, \dots, z_n). \end{tikzcd}$$

fiber $= \mathbb{C} \setminus \{z_1, \dots, z_n\} \simeq \bigvee_n S^1$  is a $K(F_n, 1)$.

LES of π_n :

$$\begin{tikzcd} \mathbb{C} - n \arrow[r] & P\mathrm{Conf}^{n+1}\mathbb{C} \arrow[r, "p"] & P\mathrm{Conf}^n \mathbb{C} \\ \pi_{\geq 2} = 0 \arrow[u] & \pi_{\geq 2} = 0. \arrow[u, squiggly] & \pi_{\geq 2} = 0 \text{ by induction hypothesis.} \arrow[u] \end{tikzcd}$$

$\square$.

Last time:

Thm: If $F \to E \xrightarrow{p} B$ is a Serre fibration, then $\exists$ LES of π_i.

pf ✓.

Prop ✓. If $F \to E \xrightarrow{p} B$ is a fiber bundle, then it is a Serre fibration. i.e. p has homotopy lifting property for all disks $\cong I^n$

pf today:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d] & E \arrow[d, "p"] \\ I^n \times I \arrow[r, "g_t"'] \arrow[ur, dashed, "\exists \tilde{g}_t"] & B \end{tikzcd}$$

$\forall n \geq 0$. p is a fiber bundle.

Goal: Find such $\tilde{g}_t$

p is a fiber bundle, $\overset{\text{Def}}{\Rightarrow}$ $\exists$ open cover $\{U_\alpha\}$ of B with local trivialization

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "h_\alpha", "\exists"'] \arrow[d, "p"'] & U_\alpha \times F \arrow[d] \\ U_\alpha \arrow[r, equal] & U_\alpha \end{tikzcd}$$

Suppose $g_t(\overset{C \times I_j}{\overset{\shortparallel}{I^n \times I}}) \subseteq U_\alpha$ for some α. (special case).

Want:

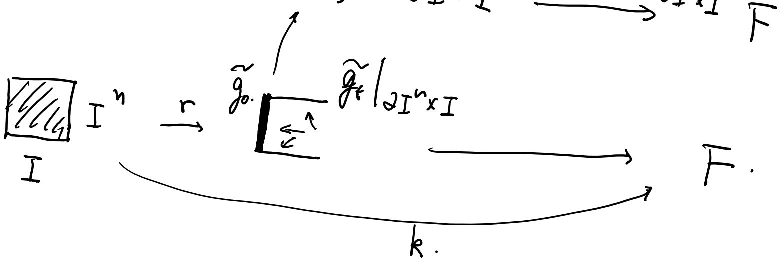
$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "\tilde{g}_0"] \arrow[d] & U_\alpha \times F \arrow[d, "p"] \\ I^n \times I \arrow[r, "g_t = G"'] \arrow[ur, dashed, "\tilde{g}_t"] & U_\alpha \end{tikzcd}$$

construct $\tilde{g}_t : \underset{\overset{\in}{x}}{I^n \times I} \longrightarrow U_\alpha \times F$. $\tilde{g}_t(x) = (g_t(x), \underset{\overset{F}{\downarrow}}{k(x)})$.

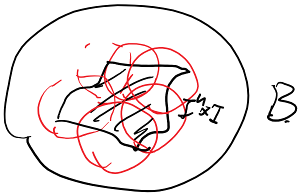
By induction on n, suppose we have a lift $\tilde{g}_t$ on $\underset{\downarrow}{\partial I^n \times I}$.

$$\begin{tikzcd} I^n\times I \arrow[r, "r"] & I^n\times\{0\}\cup\partial I^n\times I \arrow[r, "\tilde g_0\cup\tilde g_t|_{\partial I^n\times I}"] & F \end{tikzcd}$$

union of I^{k-1}



Consider the general case: $g_t(I^n\times I) \subseteq \bigcup_\alpha U_\alpha = B$.



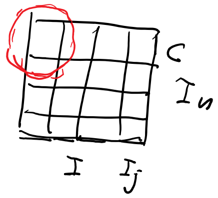
$I^n\times I$ compact $\Rightarrow$ we can subdivide I^n into a finite union of cubes C

I - - - - - - - - intervals I_j.

s.t. $G(C\times I_j)\subseteq U_\alpha$ for some α.

By ~~pro~~ special case, we have a lift $\tilde g_t$ on each $C\times I_j$.

$\rightsquigarrow$ $\tilde g_t$ on $\bigcup_{\text{finite}} C\times I_j = I^n\times I$.



□

plan for last week: Goal: "obstruction theory." (last lecture).

$\Uparrow$

- postnikov tower.
- $K(G,n)$.
- fibration.

$\uparrow$ today.

properties of fibrations.

~~Notation~~ Assume. $p: E\to B$ a fibration. B path connected.

$\forall b\in B$, write $F_b := p^{-1}(b)$ "fiber over b". well-defined up to h.e.

★ Prop. B path connected $\Rightarrow$ all fibers F_b are homotopy equivalent.

Def. Given $p_1: E_1\to B$ $p_2: E_2\to B$.

we say P

$E_1 \xrightarrow{f} E$

Def. Given $p_1 : E_1 \to B$ $\quad p_2 : E_2 \to B$.

we say $f : E_1 \to E_2$ is fiber-preserving if
$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$
or. $p_1 = p_2 f$

f is a fiber homotopy equivalence. if

$$E_1 \underset{g}{\overset{f}{\rightleftarrows}} E_2$$ s.t. fg and gf are homotopic to id through fiber-preserving maps.

Given a map $A \xrightarrow{f} B$, the pullback fibration is
$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$$f^*E := \{ (a,e) \in A \times E \mid f(a) = p(e) \} = A \underset{B}{\times} E \quad \text{"fiber product"}.$$

Prop. If $f_0, f_1 : A \to B$ are homotopic maps,

then $f_0^*E \to A$ and $f_1^*E \to A$ are fiber homotopy equivalent. (f.h.e.)

Rmk: The homotopy type of f^*E only depends on f up to homotopy.

Cor. If B is contractible, then $p : E \to B$ is f.h.e. to the trivial fibration.

$(B \to * \hookrightarrow \overset{\downarrow E}{B}.) \sim (B \xrightarrow{id} B) \Rightarrow f^*E \sim id^*E$ (with f under $B \to * \hookrightarrow B$)

"triv." $\quad \overset{\shortparallel}{E}$

$B \times F \to B$.

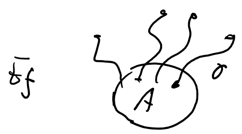
Slogan: Every map "is" a fibration.

Given a map $A \xrightarrow{f} B$,

define $E_f := \left\{ (a, \gamma) \;\middle|\; \begin{array}{l} a \in A,\ \gamma : I \to B \text{ a path} \\ \text{with } \gamma(0) = f(a) \end{array} \right\}$

Topologize: $E_f \subseteq A \times B^I$

$\uparrow$ "compact open topology".



Fact: $E_f \xrightarrow{p} B$ with $p(a,\gamma) := \gamma(1)$ is a fibration.

Note. ① $A \hookrightarrow E_f$

$a \longmapsto$ constant path on $f(a) \in B$.

② $E_f \underset{\text{h.e.}}{\simeq} A$ by shrinking each path to the start point.

" $\quad$ fibrati

② $E_f \simeq A$ by shrinking each path to the start point.

Hence,

$$\begin{tikzcd} A \arrow[r, hook, "\text{h.e.}"] \arrow[rr, bend right, "f"'] & E_f \arrow[r, "\overset{\text{fibration}}{p}"] & B \end{tikzcd}$$

Rmk: This construction is "dual" to

$$\begin{tikzcd} A \arrow[r, hook, "\overset{\text{cofibration}}{i}"] \arrow[rr, bend right, "f"'] & M_f \arrow[r, "\overset{\text{h.e.}}{r}"] & B \end{tikzcd}$$

.

any map. path connected.

Def: The homotopy fiber of $f: A \to B$ is

$F_f :=$ fiber of $E_f \xrightarrow{p} B$. (well-defined up to homotopy equivalence).

$F_f = p^{-1}(b_0) = \{(a, \gamma) \mid a \in A,\ \gamma: I \to B$ is path from $f(a)$ to $b_0\}$.

Rmk. If $p: E \to B$ is already a fibration,
then $E \hookrightarrow E_p$ is a h.e.
and $F \simeq F_p$ (homotopy fiber of p is h.e. to the fiber of p).

Ex: $A = \{b_0\} \xhookrightarrow{f} B$.

$E_f = \{\gamma: I \to B \mid \gamma(0) = b_0\} = PB$ "path space of B".

$F_f = \{\gamma: I \to B \mid \gamma(0) = b_0,\ \gamma(1) = b_0\} = \Omega B$ "loop space of B"

$\Omega B \to \underset{\text{is a point.}}{PB} \xrightarrow{p} B$. "path fibration"

LES of π_n

$\Rightarrow \quad \pi_n(B) \cong \pi_{n-1}(\Omega B) \quad \forall n$.

Ex: $\Omega K(G, n) \underset{w.}{\simeq} K(G, n-1)$

Ex:
$$\Omega K(G,n) \simeq_{w} K(G,n-1)$$
$$\begin{array}{ccc} \Omega K(\mathbb{Z},2) & \simeq_{w} & K(\mathbb{Z},1) \\ \| & & \| \\ \Omega \mathbb{C}P^{\infty} & \simeq_{w} & S^1 \end{array}$$

Similarly, $\Omega \mathbb{R}P^{\infty} \simeq_w S^0$

$$\Omega \mathbb{H}P^{\infty} \simeq_w S^3$$

Fibration

$F \to E \xrightarrow{p} B$ $\rightsquigarrow$ Fibration sequence:

↑ well defined up to h.e.

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \longrightarrow \Omega B \to F \to E \to B$$

↑ any two consecutive maps "form a fibration"

(pf skipped). e.g. $\Omega B \simeq$ homotopy fiber of $F \to E$

$\Omega E \simeq$ - - - - of $\Omega B \to F$

⋮

$F \hookrightarrow E$. homotopy fiber.

Cohomology via $K(G,n)$

notation: $[X,Y] := \{f : X \to Y\} / \text{homotopy}$.

$\langle X, Y \rangle := \{\text{based maps}\} / \text{homotopy rel base point}$.

Thm 1: There are natural bijections:

$$T : \langle X, K(G,n) \rangle \xrightarrow{\cong} H^n(X;G). \quad \forall n > 0.$$

for any CW complex X and any G = ab. gp.

Rmk: (1) What is T? Set $K := K(G,n)$.

$$H^n(K;G) \overset{UCT}{\cong} \operatorname{Hom}(\underset{\substack{\| \\ \text{Hurewicz}}}{H_n(K;\mathbb{Z})}, G) \cong \operatorname{Hom}(G,G).$$

$$\begin{tikzcd}
H^n(K;G) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(H_n(K;\mathbb{Z}), G) \arrow[r, phantom, "\cong"] & \mathrm{Hom}(G,G) \\
\alpha \arrow[u, phantom, sloped, "\in"] & \pi_n(K) \cong G \arrow[u, phantom, "\wr\ \text{Hurewicz}"] & \mathrm{id} \arrow[u, phantom, sloped, "\in"] \arrow[ll, bend left, mapsto]
\end{tikzcd}$$

$\alpha \in H^n(K(G,n);G)$ is called a "fundamental class".

$$T: \langle X, K(G,n)\rangle \longrightarrow H^n(X;G)$$
$$(f: X \to K) \longmapsto f^*\alpha$$

(2) $\langle X, K(G,n)\rangle \xrightarrow{\cong} [X, K(G,n)]$. $\forall n>0$.

easy when $n \geq 2$. since $K(G, n\geq 2)$ is 1-connected.

$n=1$. true if G is abelian (previous HW).

(3) $n=0$. $H^0(X;G) \overset{\cong}{\longleftrightarrow} [X, K(G,0)]$

$\tilde{H}^0(X;G) \overset{\cong}{\longleftrightarrow} \langle X, K(G,0)\rangle$.

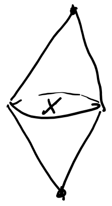
<u>pf idea of Thm 1.</u> Show that the functor $X \longmapsto h^n(X) := \underline{\langle X, K(G,n)\rangle}$ defines a (reduced) cohomology theory.

group ?

<u>Q</u>: When is $\langle Y, K\rangle$ a group?

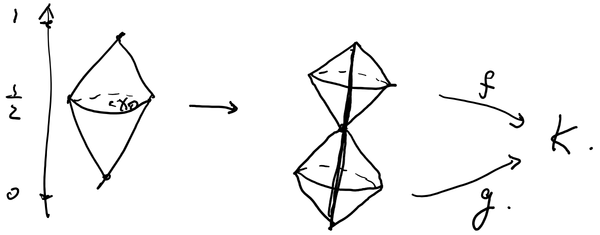
<u>A</u>: When Y is a suspension, $Y = SX$.

e.g. $Y = S^n$. $n>0$. $\langle S^n, K\rangle = \pi_n(K)$. group.



Given $f, g: SX \longrightarrow K$

define $f*g: SX \longrightarrow SX \vee SX \longrightarrow K$



We want to consider <u>based</u> maps , $x_0 \in X$ base pt.

$$\Sigma X := SX \Big/ _{\{x_0\} \times [0,1]}.$$ "reduced suspension"

<u>check</u>: $\forall$ based spaces X, K,

$$< \Sigma X , K > \text{ is a group.}$$

Lec. 29

Wednesday, June 8, 2022 1:24 PM

注：期末不考

We have: $\forall X, K$. $\langle \Sigma X, K \rangle$ is a group.

But we want: $\langle -, K \rangle$ is a group.

Use: an adjoint relation:

$$\langle \Sigma X, K \rangle \overset{\cong}{\longleftrightarrow} \langle X, \Omega K \rangle$$

↖ loop space.

$$(f: \Sigma X \to K) \longmapsto \left(\begin{array}{l} X \longrightarrow \Omega K . \\ x \longmapsto f|_{\{x\} \times [0,1]} \end{array} \right).$$

composition of loops $\rightsquigarrow$ $\Omega K \times \Omega K \longrightarrow \Omega K$.

$\longrightarrow$ $\langle X, \Omega K \rangle$ is a group.

$\langle \Sigma X, K \rangle \cong \langle X, \Omega K \rangle$ as groups.

$\langle \Sigma^n X, K \rangle \cong \langle X, \Omega^n K \rangle$ as abelian groups $n \geqslant 2$.

Def: An Ω-spectrum is a sequence of CW complexes

$K_1, K_2, K_3, \cdots$.

$$\boxed{\begin{tikzcd} X \arrow[r] \arrow[d] & PX \arrow[r] \arrow[d, phantom, "\wr *"] & \Omega X . \arrow[d] \\ Y & & \Omega Y \end{tikzcd}}$$

with a w.h.e. $K_n \xrightarrow{\sim} \Omega K_{n+1} \xrightarrow{\sim} \Omega^2 K_{n+2} \xrightarrow{\sim} \cdots$

Thm 2: If $\{K_n\}_n$ is a Ω-spectrum,

then $h^n(X) := \langle X, K_n \rangle$ defines a reduced cohomology theory.

pf skipped.

$$\begin{tikzcd} h^n(X) \arrow[r, phantom, "\underset{\text{HW.}}{\cong}"] \arrow[d, phantom, "\cong" rotate=90] & h^{n+1}(\Sigma X) \arrow[d, phantom, "\cong" rotate=90] & \\ \langle X, K_n \rangle \arrow[r, phantom, "\cong"] \arrow[rr, leftrightarrow, bend right, "K_n \xrightarrow[\text{w.h.e.}]{\sim} \Omega K_{n+1}."'] & \langle \Sigma X, K_{n+1} \rangle \arrow[r, phantom, "\overset{\text{adjoint}}{=}"] & \langle X, \Omega K_{n+1} \rangle . \end{tikzcd}$$

Example: G = ab. gp.

$K_n := K(G, n)$ CW complex.

CW approximation $\Rightarrow$ a w.h.e. $K_n \longrightarrow \Omega K_{n+1}$.

$$\begin{tikzcd} & K' \arrow[dl, shift left, "\text{h.e.}"'] \arrow[dr, "\text{w.h.e.}"] & \\ K_n \arrow[ur, shift left] \arrow[rr] & & \Omega K_{n+1} . \end{tikzcd}$$

$\{K_n\}_n$ is an Ω-spectrum.

Thm 2.

$\Longrightarrow$ $h^n(X) := \langle X, K_n \rangle$ defines a reduced cohomo. theory.

check: $h^n(X) \cong H^n(X; G)$. for X = spheres.

◎ exercise: $h^n(S^n) = h^n(\Sigma^n S^0) \underset{HW}{\cong} h^0(S^0) \cong G$.

$$\begin{tikzcd}[row sep=small] h^n(S^n) \arrow[d, phantom, "\parallel"] \\ H^n(S^n;G) \end{tikzcd}$$

$\Rightarrow$ Thm 1.

what good is $H^n(X;G) \cong \langle X, K(G,n)\rangle$?

principle: We can prove claims about $H^n(X;G)$ $\forall X$ spaces by studying the universal example $X = K(G,n)$.

Ex: $G = \mathbb{Z}$. claim: $H^1(X;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Z})$ is zero. $\forall$ space X.

$\alpha \longmapsto \alpha^2$

Pf: $H^1(X;\mathbb{Z}) \cong \langle X, K(\mathbb{Z},1)\rangle$.

$\forall \overset{\in}{\alpha}$ $\quad \overset{\in}{f}$ $\quad \overset{\parallel}{S^1}$ $\quad f: X \to S^1$

$\alpha = f^*(u)$. $\quad u \in H^1(K(\mathbb{Z},1);\mathbb{Z})$.

$\qquad \overset{\parallel}{H^1(S^1;\mathbb{Z})}$.

$\alpha^2 = (f^*u)^2 = f^*(u^2) = 0$

$u^2 \in H^2(S^1;\mathbb{Z}) = 0$. $\square$

Cup product: R = ring. $K_n := K(R,n)$

$$\begin{tikzcd} \{X \arrow[r, "f"] & K_m\} \arrow[d, leftrightarrow] & \{Y \arrow[r, "g"] & K_n\} \arrow[d, leftrightarrow] \\ & H^m(X;R) & & H^n(Y;R). \end{tikzcd}$$

cross product:

$$\begin{tikzcd} X \times Y \arrow[r, "f\times g"] \arrow[rrr, bend right] & K_m \times K_n \arrow[r] & K_m \wedge K_n \arrow[r, "\mu"] & K_{m+n}. \end{tikzcd}$$

$\in H^{m+n}(X\times Y;R)$.

what is μ ?

Hurewicz: $\pi_{m+n}(K_m \wedge K_n) \cong H_{m+n}(K_m \wedge K_n)$

$\qquad \overset{\text{is.}}{\parallel}$

$H_m(K_m) \otimes H_n(K_n)$

$\overset{\parallel}{R} \otimes \overset{\parallel}{R} \xrightarrow{\text{multiplication}} R$

$\Rightarrow \exists \mu: K_m \wedge K_n \longrightarrow K_{m+n}$. inducing $R \otimes R \to R$.

Postnikov towers.

Def: A Postnikov Tower for a path connected space X is a commutative diagram:

$$\begin{tikzcd}
 & \varprojlim X_n \arrow[d, dotted] \\
 & \vdots \arrow[d] \\
 & X_2 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur, dashed, "\exists!"] & X_1
\end{tikzcd}$$

$\rightsquigarrow$ $X \underset{\text{w.h.e.}}{\xrightarrow{\uparrow}} \varprojlim_n X_n$.

s.t. (1) $X \longrightarrow X_n$ induces iso. on π_i $\forall i \leq n$.

(2) $\pi_i(X_n) = 0$ $\forall i > n$.

~~Before~~: Note: We can assume $X_n \to X_{n-1}$ is a fibration $\forall n$.

$$F_n \to X_n \to X_{n-1}$$

LES of π_i $\Rightarrow$ $F_n = K(\pi_n X, n)$.

$$\pi_n X \overset{\shortparallel}{\;} \pi_n X_n .$$

+ (3) $X_n \to X_{n-1}$ is a fibration with fiber $= K(\pi_n X, n)$.

~~prof~~ prop. $X \longrightarrow \varprojlim_n X_n$ is a w.h.e.

pf skiped

Def. A fibration $F \to E \to B$ is principal if

$$\begin{tikzcd}
 & F \arrow[r] \arrow[d, "\simeq"] & E \arrow[r] \arrow[d, "\simeq"] & B \arrow[d, "\simeq", "\text{w.h.e.}"'] \\
\Omega B' \arrow[r] & F' \arrow[r] & E' \arrow[r] & B'
\end{tikzcd}$$

fibration sequence.

$\nwarrow$ homotopy fiber.

Rmk: F comes from $\Omega B'$ of another fibration.

Thm. A connected CW complex X has a Postnikov tower of principal fibrations iff $\pi_1 X \curvearrowright \pi_n X$ trivially $\forall n > 1$.

obstruction theory.

Consider :

(I) Extension problem :

Given a CW pair (W,A), and $A \xrightarrow{f} X$

does it extend to a map $W \to X$?

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ W \arrow[ru, dashed, "\exists ?"'] & \end{tikzcd}$$

$w_n \neq 0 \quad \forall n.$

(II) Lifting problem :

Given a fibration $X \xrightarrow{p} Y$ and a map $W \to Y$,

does it lift to a map $W \to X$?

$$\begin{tikzcd} & X \arrow[d, "p"] \\ W \arrow[ru, dashed, "\exists ?"] \arrow[r] & Y \end{tikzcd}$$

(III) Relative lifting problem .

Given (W, A) a CW pair.

a fibration $p: X \to Y$.

a map $W \to Y$.

similar construction w_n

does a partial lift $A \to X$ extend to $W \to X$?

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\ W \arrow[r] \arrow[ru, dashed, "\exists ?"] & Y \end{tikzcd}$$

Note : $A = \phi$, (III) $\Rightarrow$ (II)

$X \xrightarrow{id} X$, (III) $\Rightarrow$ (I)

Answer to (I), (II) (III) $\Leftarrow$ obstruction theory .

Consider (I) :

$$\begin{tikzcd} A \arrow[r] & X \end{tikzcd}$$

Consider (I):

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ W \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$

Suppose X has a Postnikov tower of principal fibrations.
($\exists$ iff $\pi_1 \circlearrowright \pi_n$ trivially $\forall n>1$).
$\forall n \geq 1$.

We have:

$$\begin{tikzcd} & & \varprojlim X_n . \arrow[dd, dashed] & \\ & \vdots & & \vdots \\ & & X_3 \arrow[r] \arrow[d] & K(\pi_3 X, 4) \\ & & X_2 \arrow[r] \arrow[d] & K(\pi_2 X, 3) \\ A \arrow[r] \arrow[d, hook] & X \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & X_1 \arrow[r] \arrow[d] & K(\pi_1 X, 2). \\ W \arrow[ur, dashed, "\exists"] \arrow[rr] & & X_0 = * & \end{tikzcd}$$

(if $\pi_1 X$ ab. then $\exists$ extension of Postnikov to $X_0 = *$).

Plan: Inductively on $n \geq 1$, show:

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X_n \arrow[r] \arrow[d] & PK \simeq * \arrow[d] \\ W \arrow[ur, dashed, "\exists"] \arrow[r] & X_{n-1} \arrow[r] & K. \end{tikzcd}$$

↑ induction hypothesis.

note: principal fibration:

$$\Omega K \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow K. \qquad K = K(\pi_n X, n+1).$$

check: we have a pull back fibration:

$$\begin{tikzcd} \Omega K . \arrow[r, equal] \arrow[d] & \Omega K . \arrow[d] \\ X_n \arrow[r] \arrow[d] & PK \simeq * . \arrow[d] \\ X_{n-1} \arrow[r] & K . \end{tikzcd}$$

A lift $W \to X_n$ exists $\iff$ $W \to X_{n-1} \to K$ is null homotopic.

$\Rightarrow$: $PK \simeq *$

$\Leftarrow$: constant map lifts. ✓.
homotopy lifts (HLP) ✓

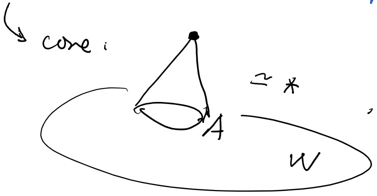
We already have a lift on $A \subseteq W$, so $A \to K$ null homotopic.

We have: $W \longrightarrow K$ with. $A \longrightarrow K$ null homotopic.

$\Longrightarrow$ a map $W \cup CA \longrightarrow K = K(\pi_n X, n+1)$.

$$\begin{tikzcd} W \arrow[r] & K \\ A \arrow[u, hook] \arrow[ur, "null"'] & \end{tikzcd}$$

cone:



$\simeq *$, $W \cup CA \simeq W/A \to K$.

$\Longrightarrow$ a class $w_n \in H^{n+1}(W \cup CA ; \pi_n X) = \langle W \cup CA, K(\pi_n X, n+1)\rangle$

$\|s$.

"obstruction class" $\to$ $w_n \in H^{n+1}(W, A ; \pi_n X)$.

prop. A lift $W \to X_n$ extending $A \to X_n$ exists iff $w_n = 0$.
pf skiped. (exercise).

Cor: If X is connected abelian CW complex. ($\pi_1 \curvearrowright \pi_n$ triv. $\forall n \geq 1$).
(W, A) CW pair,

$(*)$--- $H^{n+1}(W, A ; \pi_n X) = 0$ $\forall n$.

Then (I) always has a solution.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \\ W \arrow[ur, dashed, "\exists"] & \checkmark \end{tikzcd}$$

pf: $(*) \Rightarrow w_n = 0$ $\forall n$.

$\overset{prop.}{\Longrightarrow}$ $W \longrightarrow X_n$ $\forall n$.

$\Longrightarrow$ $W \longrightarrow \lim X$

$$\begin{tikzcd} \Rightarrow & W \arrow[r] \arrow[dr, dashed, "\exists"'] & \varprojlim X_n . \\ & & X \arrow[u, "w.h.e."'] \end{tikzcd}$$

mapping cylinder

+ compression lemma. □.

代数拓扑 1 期中考试

2022 年 4 月 22 日星期五，时长 60 分钟，满分 40 分

试卷一共四道题，请在答题本上作答。可以使用中文或者英文作答。书写须清晰可认，论述应较完整、有条理，但不要特别琐碎（时间关系）。作答时，可以直接使用我们在课上或者作业中证明或者陈述的定理，除非题目有另外的要求。

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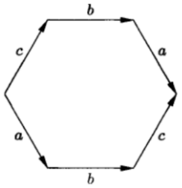
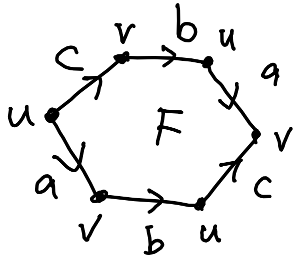
1. (10 points) Calculate all the homology groups (with $\mathbb{Z}$ coefficients) of the space obtained by gluing parallel sides of a hexagon.



2. (10 points) Let M denote the Möbius band with its boundary homeomorhic to S^1. Let X be the space obtained by attaching a disk D^2 to M by a homeomorphism of their boundaries. Compute $H_n(X;\mathbb{Z})$ for all n.
3. (10 points, 5 points each) We know that $H^4(\mathbb{CP}^2) \cong H^4(S^2 \times S^2) \cong \mathbb{Z}$. We wonder if there is a continuous map that induces such an isomorphism. Is the following statement true or false? If true, describe an example and prove that it is an example. If false, give a proof.
   (a) There is a continuous map $f : S^2 \times S^2 \to \mathbb{CP}^2$ that induces an isomorphism on $H^4(-;\mathbb{Z})$.
   (b) There is a continuous map $f : \mathbb{CP}^2 \to S^2 \times S^2$ that induces an isomorphism on $H^4(-;\mathbb{Z})$.
4. (10 points) Prove that we need at least $n+1$ contractible open subsets to cover $\mathbb{CP}^n$.

1. Use cellular homology



$C_0 = \mathbb{Z}\{u,v\}$

$C_1 = \mathbb{Z}\{a, b, c\}$

$C_2 = \mathbb{Z}\{F\}.$

$\partial_1(a) = v-u$

$\partial_1(b) = u-v$

$\partial_1(c) = v-u.$

$\partial_2(F) = 0$

$$H_0 = \frac{C_0}{Im\partial_1} = \frac{\mathbb{Z}\{u,v\}}{\mathbb{Z}\{u-v\}} \cong \mathbb{Z}$$

$$H_1 = \frac{Ker\partial_1}{Im\partial_2} = \frac{\mathbb{Z}\{a+b, b+c\}}{0} \cong \mathbb{Z}^2.$$

$$H_2 = Ker\partial_2 = \mathbb{Z}\{F\} \cong \mathbb{Z}.$$

2. Mayer-Vietoris:

$$X = M \cup_{S^1} D$$

$$\begin{tikzcd}
{} \arrow[r] & H_i(S^1) \arrow[r] \arrow[d, equal] & H_i(M) \oplus H_i(D) \arrow[r] \arrow[d, equal] & H_i(X) \arrow[r] & H_{i-1}(S^1) \arrow[r] \arrow[d, equal] & {} \\
i=1. & \mathbb{Z} \arrow[r, "1 \mapsto 2."] & \mathbb{Z} \arrow[r, two heads] & ? \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z}. \arrow[r, hook] & {} \\
 & & & \mathbb{Z}/2\mathbb{Z}. & &
\end{tikzcd}$$

$\Rightarrow H_1(X) = \mathbb{Z}/2\mathbb{Z}.$

$i=2.$

$$0 \oplus 0 \longrightarrow ? \xrightarrow{0} \mathbb{Z} \xrightarrow{1 \mapsto 2} \mathbb{Z}$$

$\Rightarrow H_2(X) = 0.$

$$H_2(X) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2\mathbb{Z} & i=1 \\ 0 & i \geq 2. \end{cases}$$

3.

(a) False.

If true, then

$$f^*: \begin{tikzcd} H^{\circ}(\mathbb{C}P^2) \arrow[r] \arrow[d, equal] & H^{\circ}(S^2 \times S^2) \arrow[d, equal] \\ \mathbb{Z}[a]/a^3=0 & \mathbb{Z}[x,y]/x^2=y^2=0. \end{tikzcd}$$

Say $f^*(a) = ix + jy \in H^2(S^2 \times S^2)$. $i, j \in \mathbb{Z}$.

then $f^*(a^2) = (f^*a)^2 = (ix+jy)^2$

$$= 2ij\, xy$$

Hence, $$f^*: \begin{tikzcd} H^4 \mathbb{C}P^2 \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^4(S^2 \times S^2) \arrow[d, phantom, "\cong" rotate=90] \\ \mathbb{Z} \arrow[r, "1 \mapsto 2ij"] & \mathbb{Z} \end{tikzcd}$$

$\mathrm{im}(f^*) \subseteq 2\mathbb{Z} \subsetneqq \mathbb{Z}$.

$\Rightarrow$ f^* is never onto.

(b) False.

If true.

$$f^*: H^{\bullet}(S^2 \times S^2) \longrightarrow H^{\bullet}(\mathbb{C}P^2).$$

$$\overset{\shortparallel}{\mathbb{Z}[x,y]/_{x^2=y^2=0}} \qquad \overset{\shortparallel}{\mathbb{Z}[a]/_{a^3=0}}.$$

Say $f^*(x) = ia$

$f^*(y) = ja \qquad i, j \in \mathbb{Z}.$

Then $\underset{\overset{\shortparallel}{0}}{f^*(x^2)} = (f^*x)^2 = (ia)^2 = i^2a^2$

$\Rightarrow i = 0.$

Similarly, $j = 0.$

So $f^*(x) = f^*(y) = 0.$

Then $f^*(xy) = f^*x \cdot f^*y = 0$

$\Rightarrow f^* = 0$ on H^4

$\square$.

4.

(a) A contractible

$\Rightarrow$ $\tilde{H}^i(X,A) \cong \tilde{H}^i(X)$ by LES of pair

Similarly for B.

Consider cup product: $i,j>0$. since $A\cup B = X$.

$$\begin{tikzcd}
H^i(X,A) \times H^j(X,B) \arrow[r, "\cup"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & H^{i+j}(X, \overset{\overset{0}{\parallel}}{A\cup B}) \arrow[d] \\
H^i(X) \times H^j(X) \arrow[r, "\cup"] & H^{i+j}(X) \\
(\alpha, \beta) \arrow[r, mapsto] & \alpha \cup \beta = 0.
\end{tikzcd}$$

$\square$.

(b) Suppose $\mathbb{C}P^n = A_1 \cup A_2 \cup \cdots \cup A_n$

each open contractible.

Consider cup products:

$$\begin{tikzcd}
\prod_{k=1}^{n} H^2(\mathbb{C}P^n, A_k) \arrow[r, "\cup"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & H^{2n}(\mathbb{C}P^n; \bigcup_{k=1}^{n} A_k) \arrow[d] \arrow[r, phantom, "="] & 0 \\
\prod_{k=1}^{n} H^2(\mathbb{C}P^n) \arrow[r, "\cup"] & H^{2n}(\mathbb{C}P^n)
\end{tikzcd}$$

$$(\alpha_1, \cdots, \alpha_n) \longmapsto \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_n = 0.$$

However, $H^{\bullet}(\mathbb{C}P^n) = \mathbb{Z}[a] / a^{n+1} = 0$.

So $\exists\, a \in H^2(\mathbb{C}P^n)$ s.t. $\underbrace{a \cup a \cup \cdots \cup a}_{n \text{ times}} \neq 0$.

contradiction.

□.