# Spectral sequences.

**Motivation**: Given a fiber bundle

$$F \to E \to B,$$

compute $H_\bullet(E)$ using info about $H_\bullet(F)$ and $H_\bullet(B)$.

**Recall**:

| | $\pi_n$ | $H_n$. |
|---|---|---|
| $A \hookrightarrow X \to X/A$<br>CW pair | ? | LES 🙂 |
| $F \to E \to B$<br>fiber bundle. | LES 🙂 | SS 😐 |

## References:

1. Hutchings, Introduction to spectral sequences.
2. Hatcher, AT Chapter 5
3. Ramos, Spectral sequences via examples.

## plan:

(I.) Spectral sequence of a filtered complex. (algebra)

(II.) Serre spectral sequences (topology).

## (I.) Algebra.

warm up: $C_* =$ a chain complex.

$F_0 C_* =$ a subcomplex of $C_*$

$\Rightarrow$ a SES of chain complexes.

$$0 \to F_0 C_* \longrightarrow C_* \longrightarrow \frac{C_*}{F_0 C_*} \longrightarrow 0 .$$

$\Longrightarrow$ LES of homology groups:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & \ker \cong \operatorname{im} \cong \operatorname{coker}\delta_{i+1} \arrow[d, dashed] & & & \\
\cdots \arrow[r, "\delta_{i+1}"] & H_i(F_0C_*) \arrow[r] & H_i(C_*) \arrow[r] \arrow[dr, dashed] & H_i\left(\frac{C_*}{F_0C_*}\right) \arrow[r, "\delta_i"] & H_{i-1}(F_0C_*) \arrow[r] & \cdots \\
 & & & \operatorname{im} \arrow[d] \arrow[r, dashed] & 0 & \\
 & & & \ker\delta_i & &
\end{tikzcd}$$

Def of $\delta$: $\forall\ \alpha \in H_i\left(\frac{C_*}{F_0C_*}\right)$,

choose $x \in C_i$ s.t. $[x] = \alpha$

$\partial x \in F_0C_{i-1}$ (or $\partial\bar{x} = 0$ in $\frac{C_{i-1}}{F_0C_{i-1}}$).

define $\delta\alpha := [\partial x]$

Suppose we want to compute $H_\bullet(C_*)$

using $H_\bullet(F_0C_*)$ and $H_\bullet\left(\frac{C_*}{F_0C_*}\right)$.

subcomplex

quotient complex

LES can be broken into SES's:

$\forall i$

$$0 \to \mathrm{Coker}(\delta_{i+1}) \to H_i(C_*) \to \ker(\delta_i) \to 0 \,..$$

<u>Summary of warm-up:</u>

To compute $H_i(C_*)$,

1. compute $H_*(\underset{\overset{\uparrow}{\text{sub}}}{F_0 C_*})$ and $H_*(\underset{\overset{\uparrow}{\text{quotient}}}{\frac{C_*}{F_0 C_*}})$.

2. Consider a 2-term chain complex:

$$H_*\left(\frac{C_*}{F_0 C_*}\right) \xrightarrow{\delta} H_{*-1}(F_0 C_*)$$

Denote homology groups by $G_0 H_* = \mathrm{coker}\,\delta$

$G_1 H_* = \ker \delta$.

3. SES

$$0 \to G_0 H_* \to H_*(C_*) \to G_1 H_* \to 0$$

$H_*(C_*)$ is determined "up to extension".

## Filtration:

A filtered R-module is an R-module $A$ with an increasing sequence of submodules

$$\cdots \subseteq F_p A \subseteq F_{p+1} A \subseteq \cdots \qquad p \in \mathbb{Z}.$$

s.t. $\bigcup_p F_p A = A$. and $\bigcap_p F_p A = 0$.

The filtration is bounded if

$$\cdots \; 0 = 0 = 0 \subseteq \underbrace{\cdots \subseteq F_p A \subseteq F_{p+1} A \subseteq \cdots}_{\text{finitely many.}} = A = A = A \cdots$$

The associated graded module is

$$G_p A := \frac{F_p A}{F_{p-1} A}.$$

we think that $\{F_pA\}_{p\in\mathbb{Z}}$ and $\{G_pA\}$ inductively "determines" A up to extension:

$$0 \to F_{p-1}A \to F_pA \to G_pA \to 0 .$$

(trivial if R = a field
manageable if R = a PID).

A <u>filtered chain complex</u> is a chain complex $(C_*, \partial)$ together with a filtration $\{F_pC_i\}_{p\in\mathbb{Z}}$ of $C_i$

s.t. $\partial(F_pC_i) \subseteq F_pC_{i-1}$.

. <u>You check</u>: $\forall p$, $(G_pC_*, \partial)$ is a chain complex

Call it <u>associated graded complex</u>.

**Note**: a filtration on $C_*$

$$\rightsquigarrow$$

a filtration on $H_i(C_*)$.

$$F_p H_i(C_*) := \left\{ \alpha \in H_i(C_*) \;\middle|\; \begin{array}{l} \exists\, x \in F_p C_* \\ \text{s.t. } [x] = \alpha \end{array} \right\}.$$

$G_p H_i(C_*) :=$ associated graded.

**Question**: How does $H_*(G_p C_*)$ determine $G_p H_*(C_*)$ ?

Recall warm-up:

$$\begin{array}{ccccc} 0 & \subseteq & F_0 C_* & \subseteq & C_* \\ \shortparallel & & & & \shortparallel \\ F_{-1} C_* & & & & F_1 C_* \end{array}$$

$$G_p H_*(C_*) = \text{homology of } \underbrace{H_*(G_1 C_*) \xrightarrow{\delta} H_*(G_0 C_*)}_{\text{2-term c.c.}}$$

When filtration has more nontrivial terms,

$E^1, E^2, \cdots,$

successive approximations.

$$H_*(G_p C_*) \rightsquigarrow G_p H_*(C_*).$$

"spectral sequence"

<u>Say</u>: spectrum and filter of light.

Suppose $(F_p C_*, \partial)$ is a filtered chain complex.

Define

$$E^0_{p,q} := G_p C_{p+q} = \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}}$$

↑

bigraded : $p =$ filtration degree

$p+q =$ total degree

$q =$ complementary degree.

$(E^0, \partial_0)$ is a chain complex with

$$\partial_0 : E^0_{p,q} \longrightarrow E^0_{p,q-1} \qquad \text{(check).}$$

Define

$$E^1_{p,q} := H_{p+q}(G_p C_*)$$

"first order approximation of $H_{\cdot}(C_*)$".

Define $\partial_1 : E^1_{p,q} \to E^1_{p-1,q}$ as:

$\forall\, \alpha \in E^1_{p,q} = H_{p+q}(G_p C_*)$

pick a chain $x \in F_p C_{p+q}$ s.t. $[x] = \alpha$

and $\partial x \in F_{p-1} C_{p+q-1}$ (since $\partial_0 \bar{x} = 0$ in $G_p C_*$)

set $\partial_1 \alpha := [\partial x]$

STOP

You check: $(E^1, \partial_1)$ is a chain complex.

Define $E^2_{p,q} := H_{p+q}(E^1_{p,q})$

$$= \frac{\operatorname{Ker}(\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q})}{\operatorname{Im}(\partial_1 : E^1_{p+1,q} \longrightarrow E^1_{p,q})}.$$

In general, for each $r = 1, 2, \ldots$ ("page number") define an "r-th order approx." to $G_p H_{p+q}(C_*)$ by

$$E^r_{p,q} = \frac{\{ x \in F_p C_{p+q} \mid \partial x \in F_{p-r} C_{p+q-1} \}}{\partial(F_{p+r-1} C_{p+q+1}) + F_{p-1} C_{p+q}}.$$

say: $$\frac{\text{"cycles up to order } r\text{"}}{\text{"boundaries up to order } r\text{"}}.$$

Lemma: Let $(F_pC_*, \partial)$ be a filtered chain complex, and define $E^r_{p,q}$ as above. Then:

(a) $\partial$ indues a well-defined map

$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r, q+r-1}$$

s.t. $\partial_r^2 = 0$

(b) $E^{r+1}$ = homology of $(E^r, \partial_r)$

i.e. $$E^{r+1}_{p,q} = \frac{\ker(\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1})}{\operatorname{Im}(\partial_r : E^r_{p+r,q-r+1} \longrightarrow E^r_{p,q})}$$

(c) $E^1_{p,q} = H_{p+q}(G_pC_*)$

(d) If the filtration of $C_i$ is bounded for each $i$,

then $\forall$ $p, q$, when $r$ is sufficiently large,

$$E^r_{p,q} = G_p H_{p+q}(C_*).$$

Pf: you. (pf straightforward but messy in notations.)

**Def:** A <u>S.S.</u> consists of

- an $R$-mod $E^r_{p,q}$, $p,q \in \mathbb{Z}$, $r \geq r_0$.
- differentials

$$\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,\, q+r-1} \text{ s.t.}$$

$\partial_r^2 = 0$

and $E^{r+1}$ = the homology of $(E^r, \partial_r)$.

A ss <u>converges</u> if $\forall\, p,q$,

if $r$ is large enough, then $\partial_r = 0$

$$\text{so } \underset{\substack{\uparrow \\ \text{call it } E^\infty_{p,q}.}}{E^r_{pq}} = E^{r+1}_{pq} = E^{r+2}_{pq} = \cdots$$

Lemma $\Rightarrow$

<u>prop</u>: If $(F_p C_*, \partial)$ is filtered complex then there is a ss $(E^r_{p,q}, \partial_r)$, $r \geq 0$ s.t. $E^1_{p,q} = H_{p+q}(G_p C_*)$.

If the filtration on $C_i$ is bounded $\forall i$, then the ss converges to

$$E^\infty_{p,q} = G_p H_{p+q}(C_*).$$

Example 1: ( Cellular homology)

$X$ = a CW complex

$X^p$ = $p$ - skeleton of $X$.

$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots$

Define $F_p C_*(X) := C_*(X^p)$

a filtration on $C_*(X)$ singular chain complex of $X$.

$$E^0_{p,q} = G_p C_*(X) = \frac{C_{p+q}(X^p)}{C_{p+q}(X^{p-1})}$$

$$E^1_{p,q} = H_{p+q}(E^0_{p,q}) = H_{p+q}(X^p, X^{p-1})$$

$$= \begin{cases} C^{cell}_p(X) & \text{if } q=0. \\ 0 & \text{else.} \end{cases}$$

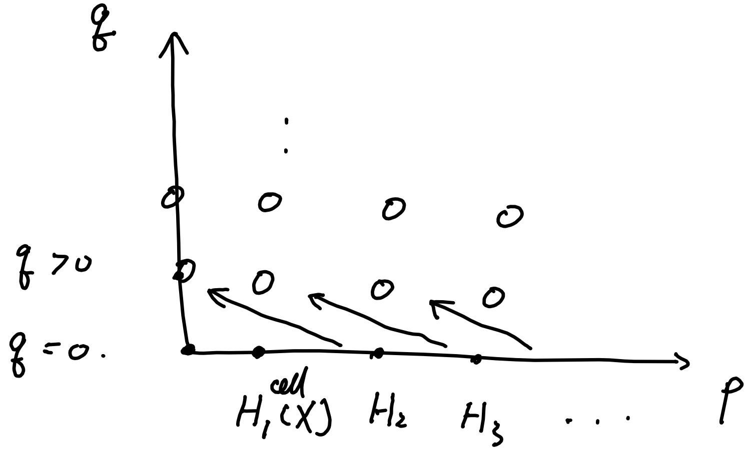
$$\begin{tikzcd}
E^1_{p,0} \arrow[r, "\partial_1"] \arrow[d, phantom, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & E^1_{p-1,0} \arrow[d, phantom, "\cong"] \\
C_p^{cell}(X) \arrow[r, "\partial^{cell}"'] & C_{p-1}^{cell}(X)
\end{tikzcd}$$

(You check).

$\hookrightarrow$ differentials in cellular chain complex

Hence, $E^2_{p,q} = \begin{cases} H_p^{cell}(X) & q=0 \\ 0 & q\neq 0. \end{cases}$

What about $E^3 = H_\cdot(E^2, \partial_2)$ ?



$\partial_2 : E^2_{p,q} \longrightarrow E^2_{p-2,q+1}$ $\quad$ $\partial_2 = 0. \Rightarrow E^3 = E^2.$

Similarly, $\partial_r = 0 \quad \forall r \geq 2$.

$$\Rightarrow \quad E^2 = E^3 = \cdots = E^\infty.$$

Suppose $X$ is fin. dim. $\Rightarrow$ filtration is bounded.

<u>prop</u> $\Rightarrow$

$$E^\infty_{p,q} = G_p H_{p+q}(C_*(X))$$

$$= G_p H_{p+q}(X).$$

we computed $\rightsquigarrow$

$$= E^2_{p,q}$$

$$= \begin{cases} H_p^{cw}(X) & q=0. \\ 0 & \text{else}. \end{cases}$$

Or,

$$G_p H_i(X) = \begin{cases} H_i^{cw}(X) & p=i \\ 0 & \text{else}. \end{cases}$$

$$\Rightarrow \quad H_i(X) = H_i^{cw}(X).$$

## (II) Apply algebra to topology:

Leray-Serre spectral seq.

**Recall:** Given a Serre fibration (has HLP for CW cmplxs) $E \xrightarrow{\pi} B$ (e.g. a fiber bundle)

if $B$ is path connected, all fibers are homotopy equivalent.

**Thm:** Let $F \to E \xrightarrow{\pi} B$ be a Serre fibration over a path. con. base $B$. Then there exists a spectral sequence

$$E^r \text{ for } r \geq 2$$

with $E^2_{p,q} = H_p(B\ ;\ H_q(F))$

and converging to

$$E^\infty_{p,q} = G_p H_{p+q}(E)$$

for some filtration on $H_\bullet(E)$.

**Rmk**: $H_\bullet(B; H_\bullet(F))$ is homology with "local coefficients" (later).

with $\pi_1 B \curvearrowright H_\bullet(F)$

Suppose $\pi_1 B = 0$. then $H_\bullet(B; H_\bullet(F))$ is homology of $B$ with coeff. in $H_\bullet(F)$.

Suppose over a field,

$$H_\bullet(B; H_\bullet(F)) \overset{UCT}{=} H_\bullet(B) \otimes_k H_\bullet(F) = H_\bullet(B \times F)$$

Leray-Serre SS:

$$\bigoplus_{p+q=i} E^2_{p,q} = H_i(B \times F)$$

$$\Downarrow$$

$$\bigoplus_{p+q=i} E^\infty_{p,q} = H_i(E).$$

$\dim H_i(E) \leq \dim H_i(B \times F)$. → trivial filtration, Pf sketch: Assume $B$ is CW complex

$$\pi_1 B = 0.$$

$\boxed{\text{STOP}}$

## (II) Apply algebra to topology:

Leray - Serre spectral seq.

**Recall**: Given a Serre fibration (has HLP for CW complex). $E \xrightarrow{\pi} B$
(e.g. a fiber bundle)

if $B$ is path connected, all fibers are homotopy equivalent.

**Thm**: Let $F \to E \xrightarrow{\pi} B$ be a Serre fibration over a 1-connected base $B$.
Then there exists a spectral sequence

$$E^r \text{ for } r \geq 2$$

with $E^2_{p,q} = H_p(B \,;\, H_q(F))$

and converging to

$$E^\infty_{p,q} = G_p H_{p+q}(E)$$

for some filtration on $H_*(E)$.

**Rmk**: $H_\bullet(B; H_\bullet(F))$ is homology with "local coefficients" (later).

with $\pi_1 B \curvearrowright H_\bullet(F)$

Suppose $\pi_1 B = 0$. then $H_\bullet(B; H_\bullet(F))$ is homology of $B$ with coeff. in $H_\bullet(F)$.

Suppose over a field,

$$H_\bullet(B; H_\bullet(F)) \overset{UCT}{=} H_\bullet(B) \otimes H_\bullet(F)$$

$$\overset{k}{=} H_\bullet(B \times F)$$

Leray-Serre SS:

$$\bigoplus_{p+q=i} E^2_{p,q} = H_i(B \times F)$$

$$\Downarrow$$

$$\bigoplus_{p+q=i} E^\infty_{p,q} = H_i(E).$$

→ trivial filtration,

$$\dim H_i(E) \leq \dim H_i(B \times F).$$

pf sketch : ( in the special case when $B$ is a CW complex).

$B^p :=$ $p$-skeleton of $B$.

Filtration on $B$ : $\cdots \subset B^p \subset B^{p+1} \subset \cdots$

$\Rightarrow$ $\cdots$ on $E$ :

$\cdots \subset \pi^{-1}(B^p) \subset \pi^{-1}(B^{p+1}) \subset \cdots$

$\Rightarrow$ Filtration on $C_*(E)$ :

$$F_p C_*(E) := C_*(\pi^{-1}(B^p))$$

$\Rightarrow \exists$ Spectral sequence of filtered complex $C_*(E)$.

$$E^0 = G_p C_*(E) = C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})).$$

$$E^1_{p,q} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})).$$

**Note:** $(B^p, B^{p-1})$ is $(p-1)$-connected

$\Rightarrow$ $(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$ is $(p-1)$-connected

$\uparrow$

LES of homotopy groups $F \to \pi^{-1}(B^p) \to B^p$

$\Rightarrow$ $E^1_{p,q} \neq 0$ only when $p \geq 0$. $q \geq 0$.

"first quadrant spectral seq."

Moreover,

$$E^1_{p,q} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$$

$$\cong.$$

$$\bigoplus_{p\text{-cells of } B} H_{p+q}(\overset{= F\times D^p}{\sigma^*(E)}, \overset{= F\times S^{p-1}}{(\sigma|_{S^{p-1}})^*(E)})$$

$\sigma : D^p \to B$

$$\begin{tikzcd} & \sigma^*E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ S^{p-1} \arrow[r, phantom, "\subseteq"] & D^p \arrow[r, "\sigma"] & B \end{tikzcd}$$

$$\cong.$$

$$\bigoplus_{p\text{-cells of } B} H_q(F). \otimes H_p(D^p, S^{p-1})$$

$$\cong$$

$$C^{cell}_p(B; H_q(F))$$

$$\partial_1 : E^1_{p,q} \longrightarrow E^1_{p-1,q}$$

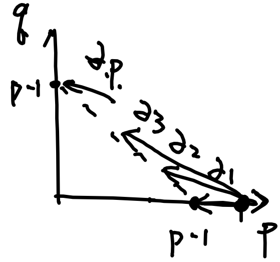
is exactly the cellular differential (you check).

$$E^2_{p,q} = \text{homology of } E^1$$

$$= H_p(B;\ H_q(F)).$$

At r-page: $\partial_r : E^r_{p,q} \longrightarrow E^r_{p-r,\,q+r-1}$

$\Rightarrow \forall p, q.\ \ \partial_r = 0 \quad \forall r > p$ ↳ $= 0$



$\Rightarrow$ spectral seq converges

$$E^r_{p,q} = E^\infty_{p,q} \quad \forall r > p.$$

By construction,

$$E^\infty_{p,q} = G_p H_{p+q}(C_*(E)) = G_p H_{p+q}(E)$$

□.

**Rmk**: If $\pi_1(B) \neq 0$.

then $\pi_1(B) \curvearrowright H_q(F)$.

The same statement holds if we let

$$H_p(B; H_q(F)) = \text{homology of } B \text{ with local coefficients in } H_q(F).$$

[ details later ].

## Examples

**Ex ①:** ( No differentials ).

Consider fibraction

$$U(1) \longrightarrow U(2) \longrightarrow S^3 .$$

[ construction: $U(2) \curvearrowright S^3 \subseteq \mathbb{C}^2$ transitively.

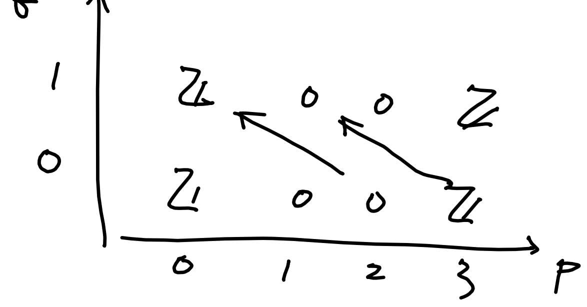
Stabilizer of $(1,0) \in \mathbb{C}^2$ is

$$\left\{ \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \lambda \end{array}\right] \,\middle|\, \lambda \in U(1) \right\} \cong U(1).$$

$\Rightarrow S^3 \cong \dfrac{U(2)}{U(1)}$. ]

LSSS:

$$E^2_{p,q} = H_p(S^3 ; H_q(S^1)) \underset{\substack{UCT \\ \pi_1 S^3 = 0.}}{\cong} H_p(S^3) \otimes H_q(S^1).$$



note: $\partial_2 = 0$.

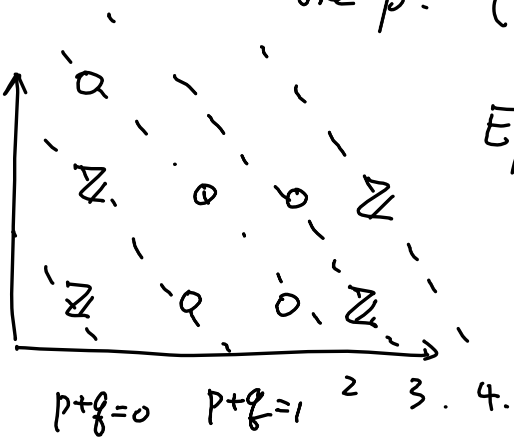
$\partial_r = 0 \quad \forall r \geq 2.$

$\Rightarrow E^2 = E^\infty$.

$$G_p H_{p+q}(U(2)) = E^\infty_{p,q}.$$

Fix $n$.

$G_p H_n(U(2))$ is nontrivial for at most one $p$. ("no extension problem").



$E^\infty_{p,n-p} = G_p H_n$

$$\Rightarrow H_n(U(2)) \cong \underset{\uparrow}{G_p} H_n(U(2))$$

~~the~~ only nontrivial one among all $p$'s.

$$= \begin{cases} \mathbb{Z} & n = 0, 1, 3, 4 \\ 0 & \text{else.} \end{cases}$$

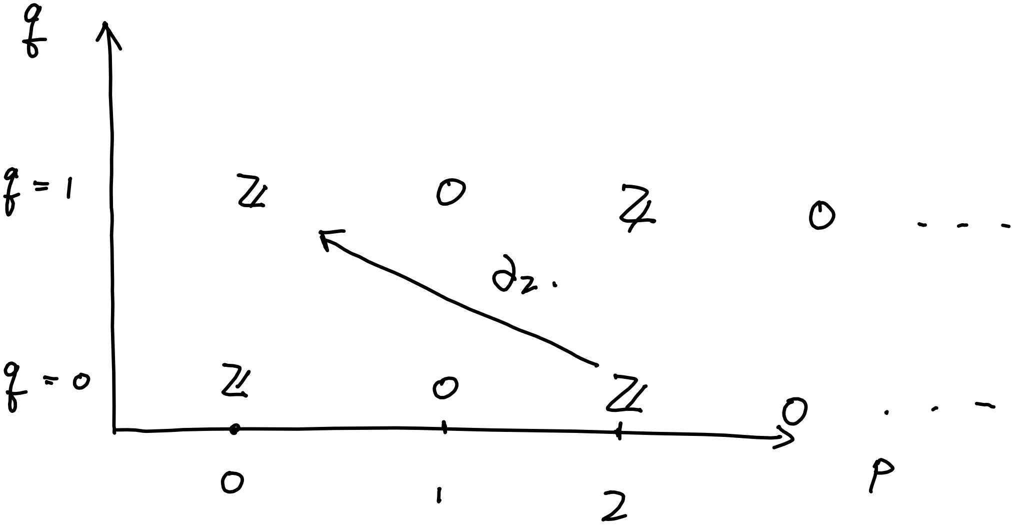
STOP

Ex1 :

$$\begin{tikzcd}
S^1 \arrow[r] & S^\infty \arrow[r] \arrow[d, phantom, sloped, "\subseteq"] & \mathbb{C}P^\infty . \\
& \mathbb{C}^\infty &
\end{tikzcd}$$

LSSS :

$$\begin{tikzcd}
E^2_{p,q} = H_p(\mathbb{C}P^\infty ; H_q(S^1)) \arrow[d, phantom, sloped, "\cong"] \arrow[d, phantom, xshift=5em, "\text{UCT}"] \\
H_p(\mathbb{C}P^\infty) \otimes H_q(S^1) .
\end{tikzcd}$$



observe: $\partial_3 = 0$ on $E^3_{p,q}$ $\forall p, q$.

$\Rightarrow$ $E^3 = E^4 = \cdots = E^\infty$

However, $E^\infty_{p,q} = G_p H_{p+q}(S^\infty) = 0$

unless $p=q=0$.

$= E^3_{p,q}$.

$\Rightarrow$ $\partial_2$ must be isomorphisms

In particular,

$$\begin{tikzcd}
\partial_2 : E^2_{2,0} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & E^2_{0,1} \arrow[d, phantom, "\cong" rotate=90] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \\
1 \arrow[r, mapsto] & 1
\end{tikzcd}$$

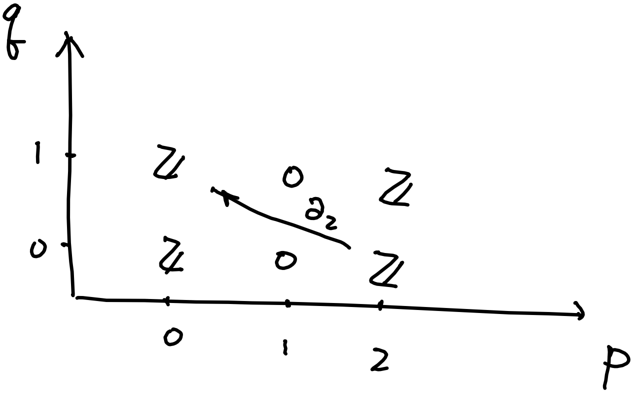
<u>Rmk</u>: In this EX, we determine $\partial_2$ by working backwards.

**Ex 2**: (Hopf fibration).

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, phantom, "\cap"] & S^3 \arrow[r] \arrow[d, phantom, "\cap"] & S^2 \arrow[d, phantom, "\shortparallel"] \\ \mathbb{C}\setminus\{0\}. & \mathbb{C}^4 & \mathbb{C}P^1 \end{tikzcd}$$

LSSS

$E^2_{p.q}$ = . —



**Claim**: $\partial_2$ is iso.

**Pf 1**: similar as before.

pf 2: (Comparison)

observe:

$$\begin{tikzcd}
S^1 \arrow[d] & S^1 \arrow[d] \\
S^3 \arrow[r, hook] \arrow[d] & S^\infty \arrow[d] \\
\mathbb{C}P^1 \arrow[r, hook, "f"] & \mathbb{C}P^\infty
\end{tikzcd}$$

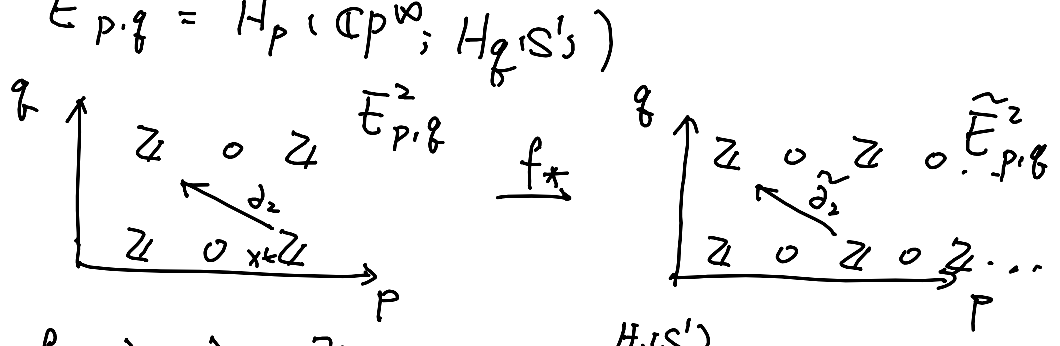
"pull back $S^1$-bundle"

LSSS is functorial:

The commutative diagram above induces:

chain maps:

$$\begin{tikzcd}
E^2_{p,q} = H_p(\mathbb{C}P^1; H_q(S^1)) \arrow[d, "f_*"] \\
\tilde{E}^2_{p,q} = H_p(\mathbb{C}P^\infty; H_q(S^1))
\end{tikzcd}$$



$f_*(\partial_2(x)) = \tilde{\partial}_2(f_*(x)) = 1 \in H_1(S^1) \Rightarrow \partial_2(x) = 1$

# Local coefficients.

$B =$ topological space , $\tilde{B} =$ univ. cov.

$\pi = \pi_1(B)$.

$M =$ a left $\mathbb{Z}[\pi]$ - module.

(so $\pi \curvearrowright M$)

Define $C_n(B;M) := C_n(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} M$.

(note: $\pi \curvearrowright \tilde{B} \Rightarrow \pi \curvearrowright C_n(\tilde{B})$

$\Rightarrow C_n(\tilde{B})$ is a (left) $\mathbb{Z}[\pi]$-module).

$\forall$ $c \in C_n(\tilde{B})$, $m \in M$. $r \in \mathbb{Z}[\pi]$.

we have $(r^{-1}c) \otimes m = c \otimes (rm)$

or $c \otimes m = (rc) \otimes (rm)$

$- \otimes_{\mathbb{Z}[\pi]} M$ is functorial $\Rightarrow$ $C_n(B;M)$ is a chain complex.

Its homology group

$$H_n(B;M)$$

is the homology of B with local coefficients in M.

"locally constant sheaf".

Similarly,

define $C^n(B;M) := \mathrm{Hom}_{\mathbb{Z}[\pi]}(C_n(\tilde{B}), M)$

a cochain complex whose cohomology

$$H^n(B;M)$$

is the cohomology of B with local. . .

Ex 1. If $\pi_1 B \curvearrowright M$ trivially,

then $C_n(B;M) = C_n(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} M \cong C_n(B) \otimes_{\mathbb{Z}} M$.

usual chain complex of B with co in M.

Then $H_n(B;M)$ = what we learned before
(trivial $M$ coefficients).

<u>Ex 2</u>: $M := \mathbb{Z}[\pi]$.

$$C_n(B;\mathbb{Z}[\pi]) = C_n(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi] = C_n(\tilde{B}).$$

$$\Rightarrow H_n(B;\mathbb{Z}[\pi]) = H_n(\tilde{B};\mathbb{Z})$$

<u>Ex 3</u>: $M := \mathbb{Z}[\pi/\pi']$ $\quad\pi'$ a subgroup of $\pi$.

$$C_n(B;\mathbb{Z}[\pi/\pi']) = C_n(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi/\pi']$$

$$= C_n(B') \quad \text{where } B' := \tilde{B}/\pi'$$

$$\begin{tikzcd} \tilde{B} \arrow[d, "\pi'"'] \arrow[dd, bend left, "\pi"] \\ B' \arrow[d] \\ B \end{tikzcd}$$

$$\Rightarrow H_n(B;\mathbb{Z}[\pi/\pi']) = H_n(B';\mathbb{Z}).$$

where $\pi_1 B' = \pi'$.

Rmk: The story is different for $H^n$.

Fact: If $B$ is a finite CW complex

$\pi = \pi_1 B$

then
$$H^n(B; \mathbb{Z}[\pi]) \cong \underset{\substack{\uparrow \\ \text{compact support.}}}{H^n_c}(\tilde{B}; \mathbb{Z})$$

Exercise: Check it on $B = S^1$.

Rmk: (Poincaré duality for nonorientable manifold)

$N :=$ a nonorie. closed $n$-mfld.

$\pi_1 N \xrightarrow{\rho} \{\pm 1\} \curvearrowright \mathbb{Z} \cong H^n(N, N \setminus pt)$.

we write

$H_k(N; \mathbb{Z}) :=$ hom. of $N$ with trivial $\mathbb{Z}$-coeff.

$H_k(N; \tilde{\mathbb{Z}}) :=$ . . . . . — — . nontrivial ~~coeff~~ $\mathbb{Z}$-coeff.

$\rho: \pi_1 N \to \{\pm 1\} \curvearrowright \mathbb{Z}$.

**Thm:** $H^k(N;\mathbb{Z}) \cong H_{n-k}(N, \tilde{\mathbb{Z}})$

$H^k(N;\tilde{\mathbb{Z}}) \cong H_{n-k}(N;\mathbb{Z})$.

Isomorphism is by cap product with a fundamental class

$$[M] \in H_n(N;\check{\mathbb{Z}})$$

## Spectral sequence for $F \to E \to B$ with $\pi_1 B \neq 0$.

Suppose $E \xrightarrow{\pi} B$ is a Serre fibration. Every path $\gamma : [0,1] \to B$ gives a homotopy equivalence

$$L_\gamma : F_{\gamma(0)} \longrightarrow F_{\gamma(1)}.$$

If $B$ path connected

then
$$\begin{tikzcd}
\pi_1(B, b) \arrow[r] & h\mathrm{Aut}(F_b) \arrow[r] & \mathrm{Aut}(H_\bullet(F_b)) \\
\gamma \arrow[r, mapsto] & L_\gamma \arrow[r, mapsto] & (L_\gamma)_*
\end{tikzcd}$$

This is called the monodromy action associated to the fibration.

<u>Leray-Serre SS:</u>

$F \to E \to B$. $B$ path-connected

$$E^2_{p,q} = H_p(B ; H_q(F))$$

homology of local coefficients.

with $\pi_1 B \curvearrowright H_q(F)$.

$\Downarrow$.

$$E^\infty_{p,q} = G_p H_{p+q}(E).$$

our construction last time:

$$E^1_{p,q} \cong C_p^{cell}(B, H_q(F))$$

$\uparrow$ with local coefficients

$\pi_1 B \curvearrowright H_q(F)$

$$\cong C_p^{cell}(\tilde{B}) \underset{\mathbb{Z}[\pi]}{\otimes} H_q(F)$$

$$= \left( \bigoplus_{\substack{p\text{-cells} \\ \text{in } B}} \mathbb{Z}[\pi] \right) \underset{\mathbb{Z}[\pi]}{\otimes} H_q(F)$$

$$= \bigoplus_{\substack{p\text{-cells} \\ \text{in } B.}} H_q(F) \cong C_p^{cell}(B) \underset{\mathbb{Z}}{\otimes} H_q(F).$$

$E^1_{p,q}$ as a module does not depend on $\pi_1 \curvearrowright H_q(F)$.

However, the differential $\partial_1$ does.

Ex 3. (Extension problem).

Consider a fibration

$$S^1 \longrightarrow U(2) \longrightarrow \mathbb{RP}^3 .$$

Construction: $U(1) \hookrightarrow U(2)$

$$\lambda \longmapsto \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

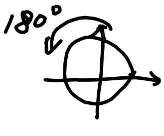
$$\frac{U(2)}{U(1)} \cong \frac{SU(2)}{\{\pm 1\}} \cong S^3/\pm 1 \cong \mathbb{RP}^3 .$$

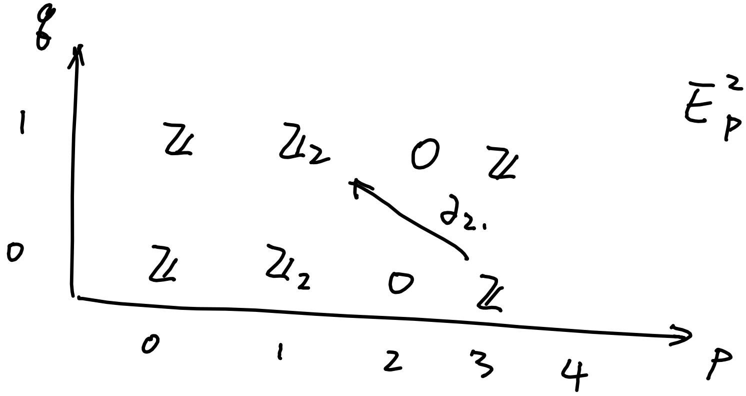
You check: $SU(2) \overset{?}{\cong} S^3 \subset \mathbb{C}^4$.

LSSS:

$$E^2_{p,q} = H_p(\mathbb{RP}^3; H_q(S^1)) \underset{\uparrow}{\cong} H_p(\mathbb{RP}^3) \otimes H_q(S^1)$$

Fact: $\pi_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$ acts trivially on $H_q(S^1)$ since $-id : S^1 \to S^1$, $\lambda \mapsto -\lambda$, induces trivial map on $H_1(S^1)$.

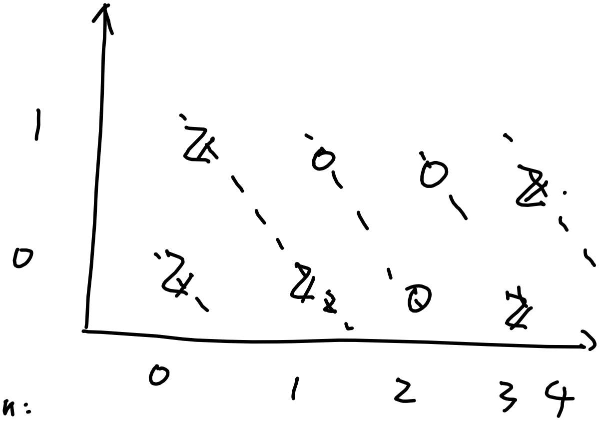




| | | | | | |
|---|---|---|---|---|---|
| 1 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | |
| 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | |
| | 0 | 1 | 2 | 3 | 4 |

$E^2_{p,q}$.

Before: $H_2(U(2)) = 0 \Rightarrow \partial_2$ is nonzero.
EXO.



| | | | | | |
|---|---|---|---|---|---|
| 1 | $\mathbb{Z}$ | 0 | 0 | $\mathbb{Z}$ | |
| 0 | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | |
| | 0 | 1 | 2 | 3 | 4 |

$E^3_{p,q} = E^\infty_{p,q}$.

p+q=n:

$E^\infty_{p,n-p} = G_p H_n(U(2))$

n=1 :

$G_1 H_1 = \mathbb{Z}_2$ , $G_0 H_1 = \mathbb{Z}$.

$$\overset{\overset{\mathbb{Z}_2}{\shortparallel}}{G_1H_1} = \frac{F_1H_1}{F_0H_1} = G_0H_1 \qquad \Rightarrow \quad 0 \to \overset{\overset{\mathbb{Z}}{\shortparallel}}{G_0H_1} \hookrightarrow \overset{\overset{?}{\shortparallel}}{F_1H_1} \twoheadrightarrow \overset{\overset{\mathbb{Z}_2}{\shortparallel}}{G_1H_1} \to 0$$

Using other method (e.g. $\bar{E}\times 0$)

we know $H_1(U(2)) \cong \mathbb{Z}$.

LES does not split

( "extension problem is nontrivial").

$E^\infty$ does <u>not</u> determine $H_\bullet$

only up to extensions.

# Spectral seq. for cohomology.

A cohomological s.s. is same as before but with arrow reversed. It consists of

- $R$-modules $E_r^{p,q}$
- differentials

$$\delta_r : E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}.$$

**Prop:** A cochain complex with a decreasing filtration $F_pC^* \supset F_{p+1}C^*$ gives a SS. $(E_r^{p,q}, \delta_r)$.

with $E_1^{p,q} = H^{p+q}(G_pC^*)$

converging to $E_\infty^{p,q} = G_p H^{p+q}(C^*)$ if fil. is bounded

Key difference : cup product.

Suppose $(C^*, \delta)$ has a product structure

$$\cup : C^i \otimes C^j \longrightarrow C^{i+j}$$

compatible with $\delta$ and filtration :

i.e.

- $\delta(\alpha \cup \beta) = (\delta\alpha) \cup \beta + (-1)^i \alpha \cup (\delta\beta)$ $\hookrightarrow \alpha \in C^i$

- $\cup : F_p C^* \otimes F_{p'} C^* \longrightarrow F_{p+p'} C^*$

Then $\cup$ induces a well-defined product on the spectral sequence :

$$\cup : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p', q+q'}.$$

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- $\cup : F_p C^* \otimes F_{p'} C^* \longrightarrow F_{p+p'} C^*$

Then $\cup$ induces a well-defined product on the spectral sequence:

$$\cup_r : E_r^{p,q} \otimes E_r^{p',q'} \longrightarrow E_r^{p+p', q+q'}.$$

<u>prop</u>: (product on $E_r^{p,q}$):

① $\delta_r$ is a derivation for $\cup_r$.

total degree $= |\alpha|$

$$\delta_r(\alpha \cup_r \beta) = (\delta_r \alpha) \cup \beta + (-1)^{p+q} \alpha \cup (\delta \beta)$$

② product on $E_{r+1}$ is

$\Rightarrow$ induced by product on $E_r$

③ . If fil. on $C^i$ is bounded $\forall i$.

product on $E_\infty^{p,q} = G_p H^{p+q}(C_*)$

is ~~the~~ equal to the product

on $G_p H^{p+q}(C^\cdot)$ induced from

the product on $H^{p+q}(C^\cdot)$

We say $E_r$ is a spectral sequence of algebras.

## Leray-Serre for cohomology:

$F \to E \xrightarrow{\pi} B$ Serre fibration

$B$ path connected.

$R =$ a commutative ring.

$H^{\bullet}(F;R)$ is a (graded commutative) $R$-algebra.

$E_2^{p,q} = H^p(B; H^q(F;R))$ is an $R$-algebra*.

Then $E_r$ is a spectral sequence of algebras converging to

$$E_\infty^{p,q} = G_p H^{p+q}(E;R).$$

Rmk (*): Let $\cup$ denote the standard product on $H^{\bullet}(B; H^{\bullet}(F;R))$

Let $\cup_2$ denote the product on $E_2$.

Then $\alpha \cup_2 \alpha' = (-1)^{qp'} \alpha \cup \alpha'$ for $\alpha \in E_2^{pq}$, $\alpha' \in E_2^{p'q'}$

The goal is to make $(E_r, \delta_r, v_r)$ a d.g.a. :

i.e ① $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$

② $\delta(\alpha\beta) = (\delta\alpha)\beta + (-1)^{|\alpha|}\alpha(\delta\beta)$

where $|\cdot| :=$ total degree.

**Ex.** ( product simplies calculation ).

**Thm** :

$H^*(SU(n)) \cong \Lambda^\cdot(a_3, a_5, \cdots, a_{2n-1})$

as an algebra.

**Pf.**: Induct on n. $n=1$ trivial.

$n \geqslant 2$ : There is a fibration

$$SU(n-1) \to SU(n) \to S^{2n-1}$$

construction: $SU(n) \curvearrowright S^{2n-1} \subseteq \mathbb{C}^n$ transitively.

Stabilizer of $\overset{\mathbb{C}^n}{\overset{\in}{(1,0,\dots,0)}}$ is

$$\left\{ \left[\begin{array}{c|c} 1 & \\ \hline & * \end{array}\right] \right\} \cong SU(n-1) \hookrightarrow SU(n)$$

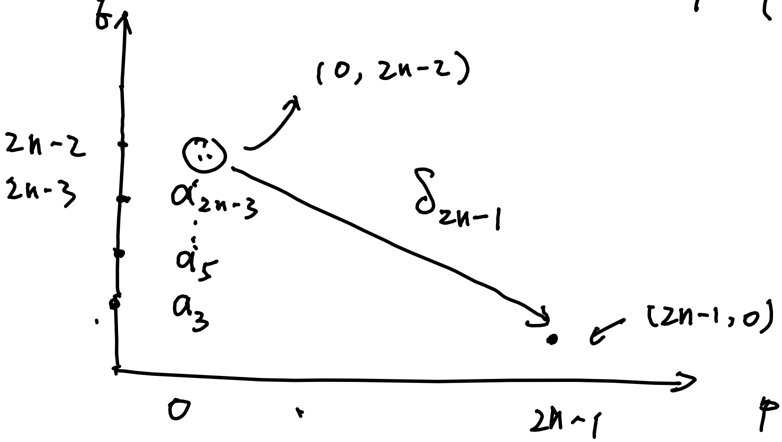
$\Rightarrow$ $S^{2n-1} \cong SU(n) / SU(n-1)$.

Consider LSSS:

$$E_2 = H^\bullet(S^{2n-1}; H^\bullet(SU(n-1)))$$

$$\cong H^\bullet(S^{2n-1}) \otimes H^\bullet(SU(n-1))$$

↑ $S^{2n-1}$ 1-connected + UCT ($H^\bullet$ has no torsion).



The only possible nonzero differential is

on $(2n-1)$-page at:

$$\delta_{2n-1} : E_{2n-1}^{0,2n-2} \longrightarrow E_{2n-1}^{2n-1,0}$$

However, we have

$$E_2 = E_3 = \cdots = E_{2n-1} \quad \text{as rings}$$

with $\delta_{2n-1} = 0$ on ring generators $a_3, a_5 \cdots a_{2n-3}$ for degree reasons.

$u \in H^{2n-1}(S^{2n-1})$

(by Leibniz rule) $\Rightarrow$ $\delta_{2n-1} = 0$ on $E_{2n-1}^{p,q}$ $\forall p,q$.

$\Rightarrow$ $E_2 = E_{2n-1} = E_{2n-2} = E_\infty$.

Finally, note

$$E_\infty = E_2 \cong H^\bullet(S^{2n-1}) \otimes \Lambda(a_3, \cdots, a_{2n-3})$$

$$\cong \Lambda^\bullet(a_3, \cdots, a_{2n-3}, a_{2n-1})$$ free abelian

$\Rightarrow$ extension problem is trivial (details skiped).

$H^\bullet(SU(n))\cong E_\infty$ as rings. $\square$

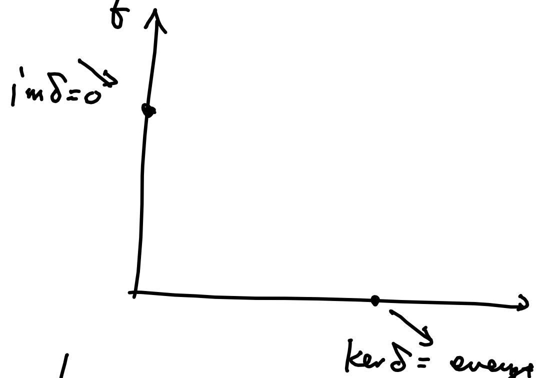
**Thm.** If $E_\infty$ is a free, graded commutative bigraded algebra,

then $H^* \cong E_\infty$ as algebras.

$$\begin{array}{ccc} H^* & \cong & E_\infty \\ \shortparallel & & \shortparallel \\ \bigoplus_i H^i & & \bigoplus_i \left(\bigoplus_p E_\infty^{p,i-p}\right) \end{array}$$

# Edge morphisms.

Suppose $E_2 \Rightarrow H$ is a first quadrant ss.



we have:

$$E_\infty^{0,q} \hookrightarrow \cdots \hookrightarrow E_3^{0,q} \hookrightarrow E_2^{0,q}$$

$$E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p,0}$$

Moreover,

$$E_\infty^{0,q} = G_0 H^q = \frac{\overset{= H^q}{F_0 H^q}}{F_1 H^q} \Rightarrow H^q \twoheadrightarrow E_\infty^{0,q}$$

$$E_\infty^{p,0} = G_p H^p = \frac{F_p H^p}{\underset{= 0}{F_{p+1} H^p}} \Rightarrow H^p \hookleftarrow E_\infty^{p,0}.$$

STOP

**Def**: The <u>edge morphisms</u> are the compositions:

$$H^q \twoheadrightarrow E_\infty^{0,q} \hookrightarrow E_2^{0,q}$$

$$E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p$$

Consider a LSSS for $F \xrightarrow{\iota} E \xrightarrow{\pi} B$.

F connected, B simply connected.

The edge morphism

$$H^q(E;R) \twoheadrightarrow E_\infty^{0,q} \hookrightarrow E_2^{0,q} = H^q(F;R).$$

is exactly $H^q(\iota)$.

$$H^p(B;R) \overset{(*)}{=} E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(E;R)$$

is exactly $H^p(\pi)$.

Example:

**Thm**: (Leray - Hirsch)

For a fibration $F \xhookrightarrow{i} E \xrightarrow{\pi} B$ with $B$ path connected.

If $i^*: H^q(E) \longrightarrow H^q(F)$ is surjective $\forall q$ and $H^*(F)$ is a finitely gen. free ab. gp, then

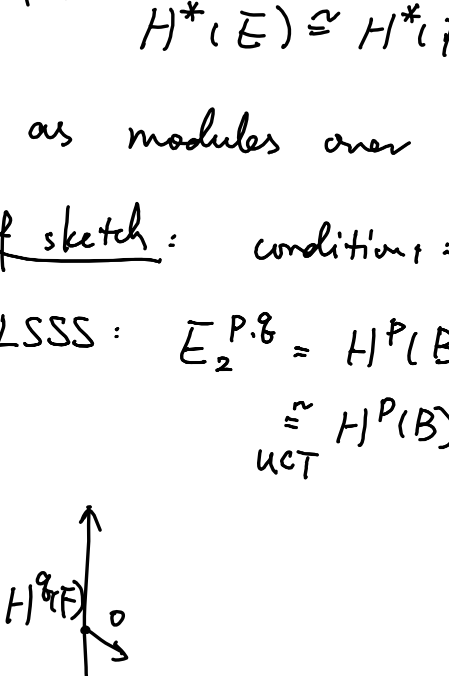
$$H^*(E) \cong H^*(F) \otimes H^*(B)$$

as modules over $H^*(B)$.

pf sketch: conditions $\Rightarrow$ $\pi_1 B \curvearrowright H^q(F)$ trivially.

LSSS: $E_2^{p,q} = H^p(B; H^q(F))$

$\underset{UCT}{\cong} H^p(B) \otimes \underset{\hookrightarrow \text{free ab.}}{H^q(F)}$.



condition 1 $\Rightarrow$ all differentials on $H^q(F)$ must be zero

$\Rightarrow E_2 \cong E_\infty$. (HW).

**Ex 2**: Fibration with a section.

$$\begin{tikzcd} \mathbb{C}\setminus n \arrow[r] & PConf^n\mathbb{C} \arrow[r, "\pi"] & PConf^{n-1}\mathbb{C} \arrow[l, dashed, bend left] \end{tikzcd}$$

$$\begin{tikzcd} E_2^{p,q} = H^p(PConf^{n-1}\mathbb{C} \,;\, H^q(\mathbb{C}\setminus n)) \arrow[d, equal] \\ H^p \otimes H^q . \end{tikzcd}$$

$$\begin{tikzcd} \text{But} \quad \pi^* : H^p(PConf^{n-1}) \arrow[r, hook] \arrow[d, equal] & H^p(PConf^n) \\ E_2^{p,0} \arrow[r, two heads] & E_\infty^{p,0} \arrow[u] \end{tikzcd}$$

$$\Rightarrow \quad E_2 = E_\infty .$$

# Vector bundles.

Refenne : § 2, 3 in Milnor- Stasheff.

**Def**: A (real) vector bundle $\xi$ over $B$ consists of the foling data:

(1) a topological space $E = E(\xi)$ "total space"

(2) projection map $\pi : E \longrightarrow B$

(3) $\forall b \in B$, vector space structure on $F_b := \pi^{-1}(b)$ $\overset{\cong}{}\ \mathbb{R}^n$.

Moreover,

$\forall b \in B$, $\exists$ neighborhood $U \subseteq B$ with a homeomorphism $h: U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ s.t. $\forall b \in U$ $h: \{b\} \times \mathbb{R}^n \longrightarrow \pi^{-1}(b)$ is an isomorphism of vector spaces.

Sometimes we write

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

fiber ↗ ($\mathbb{R}^n$) ← total space ($E$) ← base ($B$)

Examples :

1. trivial bundle $E = \mathbb{R}^n \times B$

2. tangent bundle $B = M^n$ smooth manifold

fiber at $b$ = $T_b M \cong \mathbb{R}^n$

3. normal bundle $B = M^n \subseteq W^{n+k}$ smooth manifolds.

fiber at $b$ = $T_b W / T_b M \cong \mathbb{R}^k$.

If $W$ has a Riemannian metric then

$$(T_b M)^\perp := \{ v \in T_b W \,/\, v \perp T_b M \}$$

$$\cong T_b W / T_b M .$$

so $(T_b M)^\perp \perp T_b M$.

4. $B = \mathbb{RP}^n := \{ \text{lines in } \mathbb{R}^{n+1} \}$

$$= \frac{\mathbb{R}^{n+1} \setminus \{0\}}{\mathbb{R}^\times} \cong \frac{S^{n+1}}{\{\pm 1\}}$$

Define $\gamma_n^1$ the canonical line bundle over $\mathbb{RP}^n$

as:

$$E(\gamma_n^1) := \{ (\ell, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in \ell \}$$

we have:

$$\begin{tikzcd}
\mathbb{R}^1 \cong \ell \arrow[r] & E \arrow[d, "\pi"] & (\ell, v) \arrow[d, mapsto] \\
 & \mathbb{RP}^n & \ell
\end{tikzcd}$$

**Def**: A section of $\xi$ is a map $B \xrightarrow{s} E$

s.t.
$$\begin{tikzcd} B \arrow[r, "s"] \arrow[rr, bend right, "id"'] & E \arrow[r, "\pi"] & B \end{tikzcd}$$

**Def**: $\xi$ is isomorphic to $\eta$, written $\xi \cong \eta$

if $\exists$ a homeo $f: E(\xi) \to E(\eta)$

s.t. $\forall b \in B$, $f|_{F_b(\xi)} : F_b(\xi) \xrightarrow{\cong} F_b(\eta)$

**prop**: $\forall n \geq 1$, $\gamma_n^1$ has no nonvanishing section and hence is not isomorphic to the trivial bundle.

pf: Let $s: \mathbb{R}P^n \longrightarrow E(\gamma_n')$ be a section.

Consider

$$\mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{R}P^n \xrightarrow{s} E(\gamma_n')$$

$$v \longmapsto (\langle v \rangle, t(v) \cdot v)$$

where $t(v) \in \mathbb{R}$. so $t: \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{R}$ is a cont. function.

note: $\langle v \rangle = \langle -v \rangle$

$$\Rightarrow \quad s(\langle v \rangle) = (\langle v \rangle, \ t(v)\, v)$$

$$\shortparallel$$

$$s(\langle -v \rangle) = (\langle -v \rangle, \ t(-v) \cdot (-v))$$

$$\Rightarrow \quad t(-v) = -t(v).$$

Pick any $v \in \mathbb{R}^{n+1} \setminus \{0\}$ s.t. $t(v) \neq 0$. then $t(v)$ and $t(-v)$ have different sign.

$\mathbb{R}^{n+1} \setminus 0$ is connected $\Rightarrow \exists \ v'$ s.t. $t(v') = 0$ by IVT. □.

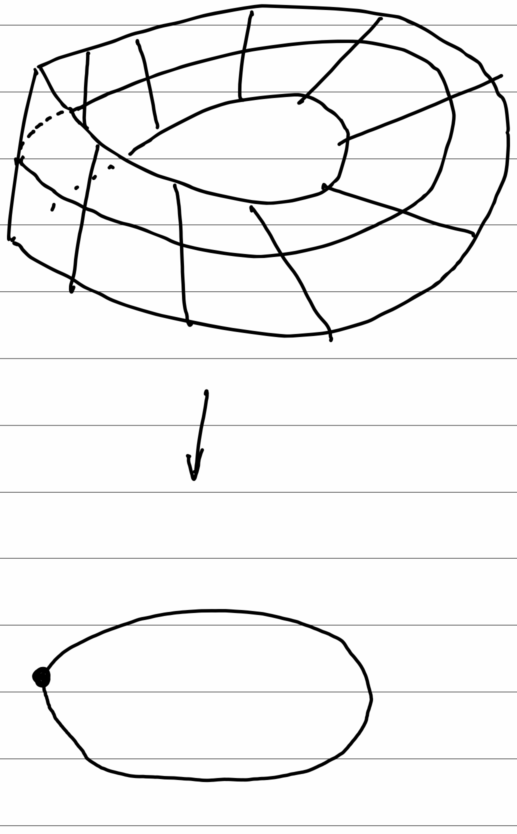
**Rmk:** when $n=1$, $\mathbb{R}P^1 \cong S^1$

$E(\gamma_1^1) \cong$ Möbius band (as $\mathbb{R}^1$-bundles over $S^1$)

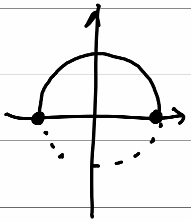
In fact,

$$\begin{tikzcd} E(\gamma_1^1) \arrow[d] \\ \mathbb{R}P^1 \end{tikzcd}$$

$\cong$



$$\mathbb{R}P^1 \cong S^1/\{\pm 1\} =$$



**Rmk**: There are two ways to think about :
a vector bundle $\xi$

① $\xi$ gives an additional structure on $B$

② $\xi$ gives a family of vector spaces $F_b$ parametrized by $b \in B$.

**Thm:** An $\mathbb{R}^n$-bundle $\xi$ is trivial iff $\xi$ has $n$ sections $s_1, \dots, s_n$ which are ~~not~~ nowhere linearly dependent.

[skip] (⇒)

Pf: $\xi$ trivial $\Rightarrow$

$$\begin{tikzcd}
B \times \mathbb{R}^n \arrow[r, "f", "\cong"'] \arrow[d] & E(\xi) \arrow[d] \\
B \arrow[u, dashed, bend left] \arrow[r, equal] & B
\end{tikzcd}$$

Let $e_1, e_2 \dots e_n$ be a basis for $\mathbb{R}^n$.

Define $s_i(x) := f((x, e_i))$

$s_1, \dots, s_n$ are nowhere dependent.

(⇐). Given $s_1, \dots, s_n$

Define $B \times \mathbb{R}^n \xrightarrow{f} E(\xi)$

$$(b, \sum t_i e_i) \longmapsto \sum t_i s_i(b).$$

check continuity on charts.

$\square$.

# Constructing new bundles from old ones.

(1) Induced bundles (or pullback).

Given ① a bundle $\xi$ :
$$\begin{tikzcd} E \arrow[d, "\pi"] \\ B \end{tikzcd}$$

② a continuous map
$$B_1 \xrightarrow{f} B$$

construct a new bundle $f^*\xi$ over $B_1$

with total space

$$E_1 := \{ (b, e) \in B_1 \times E \mid f(b) = \pi(e) \} .$$

Hence, we have
$$\begin{tikzcd} E_1 \arrow[r, "\hat{f}"] \arrow[d, "\pi_1"'] & E \arrow[d, "\pi"] \\ B_1 \arrow[r, "f"] & B \end{tikzcd}$$

Fibers: $\forall b \in B_1$, $F_b(f^*\xi) = F_{f(b)}(\xi)$

**Def** A <u>bundle map</u> from $\eta$ to $\xi$ is a continuous map $g : E(\eta) \longrightarrow E(\xi)$ that restricts to an isomorphism of vector spaces on each fiber.

**Lemma.** Given a bundle map

$$\begin{tikzcd}
E(\eta) \arrow[r, "g"] \arrow[d] & E(\xi) \\
B(\eta) \arrow[r, "\bar{g}"] & B(\xi)
\end{tikzcd}$$

Then $\eta \cong \bar{g}^* \xi$.

**Pf:** $h : E(\eta) \longrightarrow E(\bar{g}^* \xi)$

$$e \longmapsto (\pi(e), g(e)) \qquad \square$$

## (2) Cartesian product:

Given $E_1 \xrightarrow{\pi_1} B_1$ $\xi_1$

$E_2 \xrightarrow{\pi_2} B_2$ $\xi_2$

define $\xi_1 \times \xi_2$ as:

$$\pi_1 \times \pi_2 : E_1 \times E_2 \longrightarrow B_1 \times B_2 .$$

with fiber

$$(\pi_1 \times \pi_2)^{-1}(b_1, b_2) = F_{b_1}(\xi_1) \times F_{b_2}(\xi_2) .$$

## (3) Whitney sum.

Given $\xi_1$, $\xi_2$ over the same base $B$.

and $d: B \longrightarrow B \times B$ the diagonal embedding

Define $\xi_1 \oplus \xi_2 := d^*(\xi_1 \times \xi_2)$.

Fiber over $b \in B$ is

$$F_b(d^*(\xi_1 \times \xi_2)) = F_{(b,b)}(\xi_1 \times \xi_2)$$

$$= F_b \xi_1 \times F_b(\xi_2)$$

$$\cong F_b \xi_1 \oplus F_b(\xi_2).$$

More generally, operations of vector spaces gives operations of vector bundles by performing the operation fiberwise.

(check continuity, see MS §3).

e.g. $F_b(\xi_1 \otimes \xi_2) := F_b(\xi_1) \otimes F_b(\xi_2)$

$$F_b(\Lambda^k \xi) := \Lambda^k F_b(\xi)$$

$$F_b(\mathrm{Sym}^k \xi) := \mathrm{Sym}^k F_b(\xi).$$

$$F_b(\xi^\vee) := F_b(\xi)^\vee = \mathrm{Hom}_{\mathbb{R}}(F_b(\xi), \mathbb{R}).$$

...

## Euclidean vector bundles:

Take a poll.
Too fast? Too slow?

**Def**: A Euclidean vector space is a real vector space $V$ with a positive definite quadratic function

$$\mu : V \longrightarrow \mathbb{R}.$$

i.e. $\mu(\lambda v) = \lambda^2 \mu(v)$ (think $\mu(v) = |v|^2$).

$\mu(v) > 0 \quad \forall v \neq 0$.

**Rmk**: $\mu$ defines an inner product:

$$v \cdot w = \frac{1}{2}(\mu(v+w) - \mu(v) - \mu(w)).$$

**Def**: A Euclidean vector bundle is a vector bundle $\xi$ with a map

$$\mu : E(\xi) \longrightarrow \mathbb{R}$$

s.t. $\mu$ restricted to each fiber is p.d. and q.

**Rmk**: If $\xi$ is Euclidean

$\eta$ is a sub bundle of $\xi$

then $\eta \oplus \eta^{\perp} \cong \xi$.

(side: define $\eta^{\perp}$).

**Fact**. Every vector bundle over a Hausdorff, paracompact base space can be given an Euclidean metric.

(partition of unity. HW).

Recall: $B$ is paracompact if every open cover has a locally finite refinement. (assume $B$ Hausdorff)

$B$ paracompact $\Rightarrow$ $B$ has partition of unity

Ex. CW complexes, metric spaces, manifolds in $\mathbb{R}^N$.

STOP

# Stiefel - Whitney classes of vector bundles

(§4 in MS)

Plan: Define SW classes by axioms
(assuming existence).

$$\begin{tikzcd} \xi : & \mathbb{R}^n \arrow[r] & E(\xi) \arrow[d] \\ & & B(\xi) \end{tikzcd}$$

Def: The Stiefel - Whitney classes of a real vector bundle $\xi$ are

$$w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z}) \quad i = 0, 1, 2 \cdots$$

satisfying the following axioms:

A1, $w_0(\xi) = 1 \in H^0(B;\mathbb{Z}/2)$

$w_i(\xi) = 0 \quad \forall i > n.$

A2 : (Naturality) : For $f: B(\xi) \longrightarrow B(\eta)$ covered by a bundle map (so that $\xi = f^*\eta$)

$$w_i(\xi) = f^* w_i(\eta)$$

---

$\boxed{\text{side}}$:

$$\begin{tikzcd}
E(\xi) \arrow[r, "\hat{f}"] \arrow[d] & E(\eta) \arrow[d] \\
B(\xi) \arrow[r, "f"] & B(\eta)
\end{tikzcd}$$

$$\rightsquigarrow \quad H^i(B(\xi);\mathbb{Z}/2) \xleftarrow{f^*} H^i(B(\eta);\mathbb{Z}/2)$$

$$\begin{tikzcd}
f^* w_i(\eta) \arrow[d, phantom, "\parallel"] & w_i(\eta) \arrow[l, mapsto] \\
w_i(\xi).
\end{tikzcd}$$

**A3**: (Whitney product formula).

If $\xi$ and $\eta$ are vector bundles over the same base $B$,

then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta).$$

e.g. $w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta)$

$w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi)\, w_1(\eta) + w_2(\eta]$

$\vdots$

**A4**. For the canonical line bundle $\gamma_1^1$ over $\mathbb{R}P^1$ (i.e. Möbius strip).

$w_1(\gamma_1^1) \neq 0$

side: $w_1(\gamma_1^1) \in H^1(\mathbb{R}P^1; \mathbb{Z}/2) \cong \mathbb{Z}/2$

Thm (later): Such $w_i$'s exist.

## Consequences of axioms :

① If $\xi \cong \eta$, $id : B \longrightarrow B$ covered by bundle map

then $w_i(\xi) = w_i(\eta)$. (A2)

② If $\varepsilon$ is trivial

then $w_i(\varepsilon) = 0$. (A2)

③ If $\varepsilon$ is trivial, (A2+A3)

then $w_i(\varepsilon \oplus \eta) = w_i(\eta)$.

④ prop: If $\xi$ is an Euclidean $\mathbb{R}^n$- bundle with a nowhere zero section,

then $w_n(\xi) = 0$.

Pf: $\xi$ has a nonvanishing section

$\Rightarrow$ $\xi$ contains a trivial subbundle $\varepsilon$ of rank 1

Euclidean $\Rightarrow$ ~~$\oplus$~~ $\xi \cong \varepsilon \oplus \varepsilon^{\perp}$

so $\omega_n(\xi) = \omega_n(\varepsilon \oplus \varepsilon^\perp)$

$= \omega_n(\varepsilon^\perp) = 0$

since $\varepsilon^\perp$ has rank $n-1$.

⑤ prop: If an Euclidean rank-$n$ bundle $\xi$ has $k$ sections that are nowhere linearly dependent, then

$$\omega_{n-k+1}(\xi) = \omega_{n-k+2}(\xi) = \cdots = \omega_n(\xi) = 0.$$

[skip].

pf: If $s_1, \dots, s_k$ are such then they span ~~a a subbundle~~ a trivial subbundle of rank $k$ in $\xi$:

$$\varepsilon \leq \xi \Rightarrow \xi \cong \varepsilon \oplus \varepsilon^\perp$$

so $w_j(\xi) = w_j(\varepsilon^\perp) = 0$ if $j > n-k$.

$\boxed{\text{Say}}$: $\omega_{n-k+1} \neq 0 \Rightarrow \xi$ has no $k$ indep. section. $\square$.

# Total SW classes:

**Def**: The total s-w class of $\xi$ is $w(\xi) := 1 + w_1(\xi) + \cdots + w_n(\xi)$

$$H^\bullet(B(\xi);\mathbb{Z}/2) = \bigoplus_{i} H^i(B(\xi);\mathbb{Z}/2).$$

$$\cap$$

$$H^\Pi(B(\xi;\mathbb{Z}/2) := \prod_{i=0}^{\infty} H^i(B(\xi);\mathbb{Z}/2).$$

**Lemma**: The subset

$$\{w : w \text{ has leading coefficient} = 1\} \subseteq H^\Pi(B;\mathbb{Z}/2)$$

forms a commutative subgroup under multiplication.

- **Rmk**: These are precisely the group of units.

Pf: Need to check $w$ has an inverse.

say $w = 1 + w_1 + w_2 + \cdots$

$\bar{w} = 1 + \bar{w}_1 + \bar{w}_2 + \cdots$

Need: $1 = w \bar{w}$

or $\forall k \geq 1$ $\displaystyle\sum_{i=0}^{k} w_i \overline{w_{k-i}} = 0$

or $\bar{w}_k = w_1 \bar{w}_{n-1} + w_2 \bar{w}_{n-2} + \cdots +$

$w_{n-1} \bar{w}_1 + w_n$.

This gives an inductive formula for $\bar{w}_k$.

e.g. $\bar{w}_1 = w_1$

$\bar{w}_2 = w_1^2 + w_2$

$\bar{w}_3 = w_1^3 + w_3$

$\bar{w}_4 = w_1^4 + w_1^2 w_2 + w_2^2 + w_4$

$\vdots$

$\square$

**Prop**: If $\xi$ and $\eta$ are both over $B$ s.t. $\xi \oplus \eta$ is trivial then $w(\xi) = \bar{w}(\eta)$.

**Pf**: $1 = w(\xi \oplus \eta) = w(\xi) \cdot w(\eta)$ $\square$.

**Cor**: (Whitney duality)

If $M \subseteq \mathbb{R}^N$ is a smooth submanifold with $\tau$ its tangent, $\nu$ its normal

then $w(\nu) = \bar{w}(\tau)$.

**Pf**: $\mathbb{R}^N$ has a Euclidean metric.

$\tau \oplus \nu = T\mathbb{R}^N\big|_M$ = trivial of rank $N$.

$\square$

# Computing SW classes:

**Ex 1**: $\tau_{S^n} :=$ tangent bundle of $S^n$.

**prop**: $w(\tau_{S^n}) = 1$

**Pf**: , $\nu_{S^n}^{\mathbb{R}^{n+1}}$ is trivial.

$\Rightarrow w(\nu) = 1$.

$\Rightarrow w(\tau) = \bar{w}(\nu) = 1$. $\square$.

**Rmk**: Recall Poincaré - Hopf | Thurston's proof.

$\Rightarrow \nexists$ nonvanishing vector field on $S^2$

since $\chi(S^2) = 2 \neq 0$

$\Rightarrow \tau_{S^2}$ is nontrivial

Hence, it is possible that a nontrivial vector bundle to have trivial SW classes.

**Ex2**: Let $\gamma_n^1$ be the canonical (tautological) line bundle over $\mathbb{RP}^n$.

**prop**: $w(\gamma_n^1) = 1 + a$

where $a \neq 0 \in H^1(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z})$

**Recall**: $H^{\bullet}(\mathbb{RP}^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a] \Big/ (a^{n+1} = 0)$

**Pf of prop**: The inclusion $\mathbb{R}^2 \hookrightarrow \mathbb{R}^{n+1}$ gives

$$\mathbb{RP}^1 \xhookrightarrow{j} \mathbb{RP}^n$$

Moreover, we have a bundle map

$$\begin{tikzcd}
E(\gamma_1^1) \arrow[r, "\hat{j}"] \arrow[d, "\gamma_1^1"'] & E(\gamma_n^1) \arrow[d, "\gamma_n^1"] \\
\mathbb{RP}^1 \arrow[r, hook, "j"] & \mathbb{RP}^n
\end{tikzcd}$$

Hence, $H^1(\mathbb{RP}^1) \xleftarrow{j^*} H^1(\mathbb{RP}^n)$

$$w_1(\gamma_1^1) = j^* w_1(\gamma_n^1) \longmapsfrom w_1(\gamma_n^1)$$

$\downarrow$

$\neq 0$ by axiom $\quad \Rightarrow w_1(\gamma_n^1) \neq 0$

$\boxed{\text{STOP}}$

$\square$.

## Application of SW classes (to problems apparently unrelated to vector bundles].

### (I) Division algebras

**Thm** (Stiefel):

Suppose $\mathbb{R}^n$ has a bilinear product operation $p: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with no zero divisors.

Then $n$ is a power of 2.

**Ex**.:

$\mathbb{R}^1 \cong (\mathbb{R}, \times)$.

$\mathbb{R}^2 \cong (\mathbb{C}, \times)$

$\mathbb{R}^4 \cong (\mathbb{H}, \times)$ noncommutative $ij = -ji$

$\mathbb{R}^8 \cong (\mathbb{O}, \times)$ nonassociative.

In fact, these are all division algebras (Adam's thm).
(beyond our discussion).

**pf strategy**:

**Step1** If $p$ exists, then $\tau_{\mathbb{RP}^{n-1}}$ is trivial.

**Step2**: $\tau_{\mathbb{RP}^{n-1}}$ trivial $\Rightarrow$ $n$ is a power of 2.

**Lemma:** $\tau_{\mathbb{R}P^n} \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$

**Pf:** $\mathbb{R}P^n = S^n/\pm 1$

tangent bundle: $\tau_{\mathbb{R}P^n} = \tau_{S^n}/\pm 1$

$$E(\tau_{\mathbb{R}P^n}) = E(\tau_{S^n})/\pm 1$$

$$= \left\{ \pm(x, v) \in S^n \times \mathbb{R}^{n+1} \,\middle|\, \begin{matrix} |x| = 1 \\ x \cdot v = 0 \end{matrix} \right\}$$

each pair $(x, v)$ determines a function

$$f \in \mathrm{Hom}(L, L^\perp) \qquad (L = \langle x \rangle \cong \mathbb{R}^1)$$

$$f(\lambda x) = \lambda v.$$

**check:** $\tau_{\mathbb{R}P^n} \xrightarrow{\cong} \mathrm{Hom}(\gamma_n^1, \gamma^\perp)$ $\square$

**prop:** $\tau_{\mathbb{R}P^n} \oplus \varepsilon^1 \cong \underbrace{\gamma_n^1 \oplus \cdots \oplus \gamma_n^1}_{n+1}$

**pf:** $\tau \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \underset{\varepsilon^1}{\underset{\cong}{\mathrm{Hom}(\gamma_n^1, \gamma_n^1)}}$

$\cong \mathrm{Hom}(\gamma_n^1, \underbrace{\gamma^\perp \oplus \gamma_n^1}_{\overset{\parallel}{\varepsilon^{n+1}}})$

$\cong \mathrm{Hom}(\gamma_n^1, (\varepsilon^1)^{\oplus n+1})$

$\cong \mathrm{Hom}(\gamma_n^1, \varepsilon^1)^{\oplus(n+1)}$

note $\gamma_n^1 \cong \mathrm{Hom}(\gamma_n^1, \varepsilon)$ by picking a metric on $\gamma_n^1$

$\cong (\gamma_n^1)^{\oplus(n+1)}$ $\square$.

**Cor** (Stiefel): $w(\tau_{\mathbb{R}P^n}) = 1 \iff$ iff $(n+1)$ is a power of 2.

Hence, $\tau_{\mathbb{R}P^n}$ is nontrivial when $n \neq 2^k - 1$.

证: $\omega(\tau_{\mathbb{R}P^n}) = \omega(\tau \oplus \varepsilon^1)$

$= \omega((\gamma_n^1)^{\oplus n+1}) = \omega(\gamma_n^1)^{(n+1)}$

$= (1+a)^{n+1}$

$= 1 + \sum_{i=1}^{n} \binom{n+1}{i} a^i \qquad (a^{n+1}=0)$

$= 1$ iff $n+1$ is a power of 2. □.

**Pf of Thm** : Given $p: \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

**note**: every $z \neq 0$ in $\mathbb{R}^n$ gives an iso $\mathbb{R}^n \xrightarrow{\cdot z} \mathbb{R}^n$

$$y \longmapsto p(y,z) \quad \forall y$$

Pick basis $e_1, \dots, e_n$ for $\mathbb{R}^n$.

$\exists$ isomorphism $v_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

s.t. $v_i(p(y,e_1)) = p(y,e_i) \quad \forall i=1,\dots,n$

For any line $L \subseteq \mathbb{R}^n$, define

$\bar{v}_i : L \longrightarrow L^\perp \qquad \mathbb{R}^n \longrightarrow L^\perp$

by $\bar{v}_i(x) :=$ image of $v_i(x)$ under projection

$\bar{v}_i$ gives a section of $\mathrm{Hom}(\gamma^1_{n-1}, \gamma^\perp) \cong \tau_{\mathbb{RP}^{n-1}}$

**check**: $\bar{v}_1 = 0$, $\bar{v}_2 \cdots \bar{v}_n$ are independent $\forall L$.

Hence, $\tau_{\mathbb{RP}^{n-1}}$ is trivial.

$\Rightarrow$ n is a power of 2 $\square$.

## (II) SW classes as obstruction to immersions.

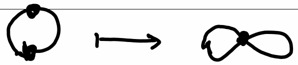
**prop.** If $M^n$ can be immersed in $\mathbb{R}^{n+k}$

then $\bar{w}_i(M) = 0 \quad \forall i > k$.

**Recall:** $f : M \to \mathbb{R}^{n+k}$ is an immersion if

$T_pf : T_pM \to T_{f(p)}\mathbb{R}^{n+k}$ is injective.

e.g. $S^1 \hookrightarrow \mathbb{R}^2$



**pf of prop.** $M \hookrightarrow \mathbb{R}^{n+k}$

$$\Rightarrow \tau_M \oplus \nu_M = \varepsilon^{n+k}$$

$$\Rightarrow w_i(\nu) = \bar{w}_i(\tau_M) \overset{\text{def}}{=} \bar{w}_i(M) \quad \forall i$$

$\dim \nu = k \quad \Rightarrow \quad \bar{w}_i(M) = w_i(\nu) = 0 \quad \forall i > k.$

$\square$.

**Ex**: When $n = 2^r$, if $\mathbb{R}P^n \hookrightarrow \mathbb{R}^{n+k}$,

then $k \geq n-1$.

$\omega(\mathbb{R}P^n) = (1+a)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k = 1+a+a^n$
when $n=2^r$

$\bar{\omega}(\mathbb{R}P^n) = 1+a+a^2+ \cdots + a^{n-1}$

$\xRightarrow{\text{prop}}$ $k \geq n-1$.

When $n=2^r$, $\mathbb{R}P^n$ cannot be immersed in $\mathbb{R}^{2n-2}$.

**Recall**:

**Thm** (Whitney): If $M^n$ smooth compact $n>1$
then $M^n$ can be immersed in $\mathbb{R}^{2n-1}$.

Ex $\Rightarrow$ Thm is sharp!

## (III) Stiefel - Whitney numbers and cobordism.

Consider the base $B = M^n$

a closed smooth $n$-manifold

(possibly disconnected).

$w_i(M) := w_i(\tau_M)$ for $i = 0 \cdots n$.

Consider a monomial

$$w_1(M)^{r_1} w_2(M)^{r_2} \cdots w_n(M)^{r_n} \in H^{D}(M;\mathbb{Z}/2)$$

where $D = \sum_{i=1}^{n} i\, r_i$

When $D = n$, we get numbers (mod 2)

$$\langle w_1(M)^{r_1} w_2(M)^{r_2} \cdots w_n(M)^{r_n}, [M] \rangle \overset{\mathbb{Z}/2\mathbb{Z}}{\in}$$

called the Stiefel-Whitney number of $M$

associated to the monomial $w_1^{r_1} \cdots w_n^{r_n}$.

**Thm.** (Pontrjagin).

If $M$ is the boundary of a smooth compact $(n+1)$-dim. manifold $V$, then all the SW numbers of $M$ are zero.

$\boxed{\text{STOP}}$

Pf. $M = \partial V$.

$$H_{n+1}(V, \partial V) \xrightarrow{\partial} H_n(\partial V)$$

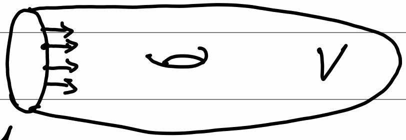
$$[V] \longmapsto \partial[V] = [M]$$

dually, $H^{n+1}(V, \partial V) \xleftarrow{\delta} H^n(\partial V) \ni \alpha$ $\forall \alpha$.

we have $\langle \delta\alpha, [V] \rangle = \langle \alpha, \partial[V] \rangle = \langle \alpha, [M] \rangle$ (*)

Consider $\tau_V|_M = i^* \tau_V$ where $i : M \hookrightarrow V$.

observe: $\nu_M^V = \varepsilon$ trivial.



$\partial V = M$

$$\tau_V\big|_M = \tau_M \oplus \nu_M^V$$

$$= \tau_M \oplus \varepsilon$$

$$\Rightarrow \quad w_k(\tau_V\big|_M) = w_k(\tau_M) \qquad \forall k.$$

OTOH: $LHS = w_k(i^* \tau_V)$

$$= i^* w_k(\tau_V).$$

$$\Rightarrow \quad w_k(\tau_M) = i^* w_k(\tau_V) \qquad \forall k.$$

Consider LES:

$$H^n(V) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(V, M).$$

Stiefel - W #:

$$\langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}, [M] \rangle$$

$$\overset{\text{by}(*)}{=} \langle \delta\left(w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}\right), [V] \rangle.$$

$$= \langle \underbrace{\delta \, (i^*}_{\overset{\shortparallel}{0}.}(w_1(\tau_V)^{r_1} \cdots w_n(\tau_V)^{r_n})), [V] \rangle$$

$$= 0. \qquad \square$$

**Thm** (Thom).

If all SW numbers of $M$ are zero. then $M$ can be realised as ~~the~~ $M = \partial V$ for some $V^{n+1}$ smooth compact.

**Def**: Two smooth closed $n$-manifolds $M_1$ and $M_2$ belong to the same cobordism class iff $M_1 \amalg M_2$ is the boundary of a smooth compact $(n+1)$-manifold.

e.g.

**Cor**: $M_1$ and $M_2$ belong to the same cobordism class $\Leftrightarrow$ they have the same SW numbers.

(apply Thm to $M_1 \amalg M_2$)

## Summary:

SW are invariants of $\mathbb{R}$-vector bundles.
They are useful tools for proving certain objects cannot exist.

$\boxed{\text{STOP}}$

# Universal bundle

**Goal**: For each $n$, construct a universal bundle $\gamma^n$:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r] & E(\gamma^n) \arrow[d] \\
 & B(\gamma^n)
\end{tikzcd}$$

s.t.

**Thm.** Suppose $B$ is a paracompact space.

(1) Any $\mathbb{R}^n$-bundle $\xi$ over $B$ admits a bundle map $\xi \to \gamma^n$.

(2) Any two bundle maps $f, g : \xi \to \gamma^n$ are bundle-homotopic.

i.e. $\exists\ h_t : \xi \to \gamma^n$, $t \in [0,1]$

s.t. $h_0 = f$, $h_1 = g$.

## Consequence:

(1) Any $\mathbb{R}^n$- bundle $\xi$ is a pullback of $\gamma^n$.

(Recall:

$$\begin{tikzcd} E(\xi) \arrow[r, "\tilde{f}"] \arrow[d] & E(\gamma^n) \arrow[d] \\ B \arrow[r, "f"] & B(\gamma^n) \end{tikzcd}$$

is a bundle map

$\Rightarrow$ $\xi = f^* \gamma^n$.)

(2) Universal bundle is unique. (up to homotopy equivalence).

If $\gamma^n$ and $\delta^n$ both satisfy this,

then $\exists$ $\gamma^n \underset{g}{\overset{f}{\rightleftarrows}} \delta^n$

$f \circ g \simeq id_{\gamma^n}$ and $g \circ f \simeq id_{\delta^n}$.

(3) There is a bijection

$$\{ f : B \to B(\gamma^n) \} \Big/ \text{htpy} \overset{\simeq}{\longleftrightarrow} \{ \mathbb{R}^n \text{ bundles over } B \} \Big/ \text{iso.}$$

$f \longmapsto f^* \gamma^n.$

(4) A characteristic class of $\mathbb{R}^n$-bundle is a natural way to assign

$$\xi \text{ over } B(\xi) \rightsquigarrow c(\xi) \in H^{\bullet}(B(\xi))$$

s.t. $c(f^* \xi) = f^* c(\xi)$.

Claim: There is a bijection

$$\left\{ \begin{matrix} \text{characteristic} \\ \text{classes of} \\ \mathbb{R}^n\text{-bundles} \end{matrix} \right\} \longleftrightarrow H^{\bullet}(B(\gamma^n))$$

$$c \longmapsto c(\gamma^n)$$

$$f^*\alpha \longleftarrow\!\shortmid \alpha$$

where

$$\begin{tikzcd}
E(\xi) \arrow[r, "\tilde{f}"] \arrow[d] & E(\gamma^n) \arrow[d] \\
B(\xi) \arrow[r, "f"] & B(\gamma^n)
\end{tikzcd}$$

## Construction of $\gamma^n$

~~$V_{n,k}$~~. For $n, k$ ~~pos~~ $\in \mathbb{N}$.

**Def**: The Grassmannian is

$$G_n(\mathbb{R}^{n+k}) := \{ H \mid H \text{ is an } n\text{-dimensional linear subspace of } \mathbb{R}^{n+k} \}$$

**Rmk**:

① $GL_{n+k}(\mathbb{R}) \curvearrowright G_n(\mathbb{R}^{n+k})$ transitively.

Fix $H = \langle e_1, \cdots, e_n \rangle \leq \mathbb{R}^{n+k}$ (so $H \in G_n(\mathbb{R}^{n+k})$)

$$G_n(\mathbb{R}^{n+k}) = GL_{n+k}(\mathbb{R}) \Big/ \mathrm{Stab}_{GL}(H)$$

$$\mathrm{Stab}_{GL}(H) = \left\{ \begin{array}{c} n \\ k \end{array} \left[ \begin{array}{c|c} * & * \\ \hline 0 & * \end{array} \right] \begin{array}{c} n \\ k \end{array} \right\}$$ closed Lie subgroup.

Hence $G_n(\mathbb{R}^{n+k})$ is a smooth manifold of $\dim = nk$ (Not a Lie gp).

② In fact,

$$\begin{array}{l} \cdot \quad O_{n+k} \circlearrowright G_n(\mathbb{R}^{n+k}) \\ \quad\ \ \cap| \\ \quad GL_{n+k} \end{array}$$

$$\mathrm{Stab}_{O_{n+k}}(H) = \left\{ \left[\begin{array}{c|c} A & \\ \hline & B \end{array}\right] : \begin{array}{l} A \in O_n \\ B \in O_k \end{array} \right\}$$

$$= O_n \times O_k \le O_{n+k}.$$

Thus, $G_n(\mathbb{R}^{n+k}) \cong O_{n+k} / O_n \times O_k$ is a ~~smooth~~ compact manifold of dim $nk$.

③ $$G_n(\mathbb{R}^{n+k}) \cong G_k(\mathbb{R}^{n+k})$$

$$H \longmapsto H^{\perp}$$

④ (n=1) $$G_1(\mathbb{R}^{1+k}) = \mathbb{R}P^k$$

The canonical bundle $\gamma^n_k$ over $G_n(\mathbb{R}^{n+k})$

is :

$$\begin{tikzcd} H \arrow[r] & E(\gamma^n_k) := \{(H, v) \in G_n(\mathbb{R}^{n+k}) \times \mathbb{R}^{n+k} : v \in H\} \arrow[d] \\ & G_n(\mathbb{R}^{n+k}) \end{tikzcd}$$

$$\begin{tikzcd} (H, v) \arrow[d, mapsto] \\ H \end{tikzcd}$$

<u>Rmk</u>: This generalises $\gamma^1_k$ over $\mathbb{RP}^k$ $(n=1)$.

<u>Observe</u>. Fix $n$.

$\mathbb{R}^n \subseteq \mathbb{R}^{n+1} \subseteq \mathbb{R}^{n+2} \subseteq \cdots$ $\qquad$ $\mathbb{R}^\infty := \bigcup_{k=0}^{\infty} \mathbb{R}^{n+k}$

$\rightsquigarrow$

$G_n(\mathbb{R}^n) \subseteq G_n(\mathbb{R}^{n+1}) \subseteq \cdots$ $\qquad$ $G_n(\mathbb{R}^\infty) := \bigcup_{k=0}^{\infty} G_n(\mathbb{R}^{n+k})$

Define $\gamma^n = \gamma^n_\infty$ as :

$$\begin{tikzcd} H \arrow[r] & E(\gamma^n) := \{ (H, v) \in G_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty : v \in H \} \arrow[d] \\ & G_n(\mathbb{R}^\infty) =: G_n \end{tikzcd}$$

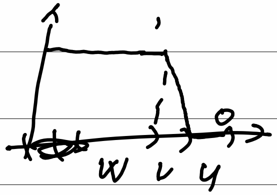
says: $\gamma^n$ is the universal $\mathbb{R}^n$-bundle.

Make a claim first : $\hat{f}$.

pf of Thm (1) : Choose open cover $\{U_i\}_{i \in I}$ of $B$ s.t. $\xi|_{U_i}$ is trivial.

$B$ paracompact $\Rightarrow$ we ~~cho~~ can make the cover countable and locally finite,

pick a partition of unity $\{\lambda_i\}_{i \in I}$ subordinate to $\{U_i\}_{i \in I}$, s.t.



① $\lambda_i : B \to \mathbb{R}$

② we have open $W_i \subseteq V_i \subseteq U_i$ ($\{V_i\}_{i \in I}$) still cover

s.t. $\overline{W_i} \subseteq V_i$ . $\overline{V_i} \subseteq U_i$

$\lambda_i = 1$ on $\overline{W_i}$ and $\lambda_i = 0$ outside $V_i$

③ $\forall x \in B(\xi)$, $\sum_{i \in I} \lambda_i(x) = 1$.

(Not needed).

$\xi|_{U_i}$ trivial $\Rightarrow$

$$\begin{tikzcd} \pi^{-1}U_i \arrow[r, "\cong"] \arrow[rr, bend right, "h_i"'] & U_i \times \mathbb{R}^n \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

Define $h_i' : E(\xi) \longrightarrow \mathbb{R}^n$

$h_i'(e) = 0$ if $\pi(e) \notin V_i$

$h_i'(e) = \lambda_i(\pi(e))\, h_i(e)$ if $\pi(e) \in U_i$

note: $h_i'$ is cont. (supported on $\overline{V_i}$)..

Define $\hat{f} : E(\xi) \longrightarrow \mathbb{R}^\infty$ by

$$\hat{f}(e) = (h_i'(e))_{i=0}^{\infty} \in (\mathbb{R}^n)^\infty \longrightarrow \mathbb{R}^\infty.$$

since $\{U_i\}_i$ is locally finite.

Key property: $\hat{f}$ is injective and ~~to~~ linear on each fiber of $\xi$.

Define $f : E(\xi) \longrightarrow E(\gamma^n)$

$$e \longmapsto \left( \hat{f}(F_{\pi(e)}(\xi)) , \hat{f}(e) \right).$$

is a bundle map. ($\hat{f}$ is ~~linear~~ injective and linear on fiber).

$\square$.

For uniqueness, say $\xi \overset{f}{\underset{g}{\rightrightarrows}} \gamma^n_\infty$.

each gives $\hat{f}, \hat{g} : E(\xi) \longrightarrow \mathbb{R}^\infty$

linear and injective on fibers.

Define $\hat{h}_t(e) = (1-t)\hat{f}(e) + t\hat{g}(e)$ $\quad 0 \leq t \leq 1$

$\hat{h}_t$ is injective if $\hat{f}(e)$ and $\hat{g}(e)$ indep. $\forall e$.

In general, consider $d_{\text{even}} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$

$$e_i \longmapsto e_{2i}$$

$$d_{\text{odd}} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$$

$$e_i \longmapsto e_{2i-1}.$$

Then $f \sim d_1 \circ f \sim d_2 \circ g \sim g$. $\square$

**Goal:**

$$H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}/2) = \mathbb{Z}_2[w_1, \dots, w_n]$$

↑ SW classes

**Cell structure for $G_n(\mathbb{R}^m)$:**

Fix a "flag":

$$\mathbb{R}^0 \subseteq \mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots \subseteq \mathbb{R}^m.$$

Any $X \in G_n(\mathbb{R}^m)$ gives a sequence:

$$0 \leq \underbrace{\dim(X \cap \mathbb{R}^1) \leq \dim(X \cap \mathbb{R}^2)}_{\text{difference} \leq 1.} \leq \cdots \leq \dim(X \cap \mathbb{R}^m) \overset{\shortparallel}{} n$$

say: "a sequence of length m with n jumps".

**Def:** A Schubert symbol $\sigma = (\sigma_1, \dots, \sigma_n)$ is a sequence of integers

$$1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_n \leq m.$$

$e(\sigma) := \{ X \in G_n(\mathbb{R}^m) :$

$\dim(X \cap \mathbb{R}^{\sigma_i}) = i$ and $\dim(X \cap \mathbb{R}^{\sigma_i - 1}) = i-1 \}$.

say: $\sigma_i$'s are places of jumps.

Thm: $\{ e(\sigma) : \sigma$ is a Schubert symbol$\}$ form a CW-complex with underlying space $G_n(\mathbb{R}^m)$.

Let $H^k \subseteq \mathbb{R}^k$ be the upper half plane with $x_k > 0$.

Lemma: each $X \in e(\sigma)$ has a unique orthonormal basis $(x_1, \dots, x_n) \in H^{\sigma_1} \times \cdots \times H^{\sigma_n}$

Pf: $\dim X \cap \mathbb{R}^{\sigma_1} = 1$ . . .
$\dim X \cap \mathbb{R}^{\sigma_2} = 2$. . . $\square$.

Cor: $\dim e(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$.

pf sketch:

Let $e'(\sigma) := \left\{ (x_1, \dots, x_n) \;\middle|\; x_i\text{'s are orthonormal, } x_i \in H^{\sigma_i} \ \forall i \right\}$

$\bar{e}'(\sigma) := \left\{ (x_1, \dots, x_n) \;\middle|\; \text{orthonormal, } x_i \in \overline{H}^{\sigma_i} \ \forall i \right\}$.

By induction on $n$, show that

$\bar{e}'(\sigma) \cong$ closed ball of dim $\boxed{\text{STOP}}$

$d(\sigma) = \sum_i (\sigma_i - i)$

e.g. $n=1$. $\bar{e}'(\sigma_1) = \{ x_1 \mid x_1 = (x_{11}, \dots, x_{1\sigma_1}, 0 \dots 0),$
$|x_1| = 1. \ x_{1\sigma_1} \geq 0 \} \cong D^{\sigma_1 - 1}$.

The map

$$\bar{e}'(\sigma) \longrightarrow G_n(\mathbb{R}^m)$$
$$(x_1, \dots, x_n) \longmapsto \operatorname{span}\{x_1, \dots, x_n\}.$$

takes $e'(\sigma)$ homeomorphically onto $e(\sigma)$.

Check this is a CW complex ... [see MS §6].

Hence, $G_n(\mathbb{R}^m)$ is a finite CW complex with $\binom{m}{n}$ cells

Take $m \to \infty$,

$G_n(\mathbb{R}^\infty)$ is an infinite CW complex

<u>Q</u>: How many $r$-cells are there in $G_n(\mathbb{R}^m)$ ?

Suppose $e(\sigma)$ has dim

$$\dim = \sum_{i=1}^{n} \underbrace{(\sigma_i - i)}_{\mu_i} = r$$

so we have $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq m-n$

$$\mu_1 + \cdots + \mu_n = r$$

<u>Cor</u>: number of $r$-cells in $G_n(\mathbb{R}^m)$ = number of partitions of $r$ into a sum of at most $n$ positive integers $\leq m-n$.

**Thm**: $H^*(G_n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1, \cdots, w_n]$.

where $w_k = w_k(\gamma^n)$.

**Pf**: We will construct an isomorphism.

$$H^*(G_n; \mathbb{Z}/2) \xrightarrow{f^*} \mathbb{Z}/2\mathbb{Z}[s_1, \cdots, s_n]$$

$$w_k(\gamma^n) \longmapsto s_k.$$

skip for now

Consider $\gamma^1$ over $G_1 = \mathbb{RP}^\infty$.

HW $\Rightarrow$ $w(\gamma^1) = 1 + a$    $\deg a = 1$.

where $H^*(\mathbb{RP}^\infty; \mathbb{Z}/2) = \mathbb{Z}/2[a]$.

Consider $(\gamma^1)^{\times n}$ over $(\mathbb{RP}^\infty)^{\times n}$

HW $\Rightarrow$ $w((\gamma^1)^{\times n}) = (1+a_1) \times (1+a_2) \cdots (1+a_n)$

where $H^*((\mathbb{RP}^\infty)^{\times n}) = \mathbb{Z}/2[a_1, a_2, \cdots, a_n]$.

$\deg a_i = 1$ $\forall i$.

$\Rightarrow \forall k, \quad w_k((\gamma^1)^n) = s_k(a_1, \cdots, a_n)$

where $s_k$ is the k-th elementary sy. poly.

e.g. $s_0 = 1$

$s_1 = a_1 + \cdots + a_n$

$s_2 = \sum_{i<j} a_i a_j$

$\vdots$

Consider the classifying map $f$ of $(\gamma^1)^{\times n}$.

i.e. $f : (\mathbb{R}P^\infty)^{\times n} \longrightarrow G_n$

s.t. $(\gamma^1)^{\times n} = f^* \gamma^n$

We have $f^* : H^\bullet(G_n) \longrightarrow H^\bullet((\mathbb{R}P^\infty)^{\times n})$.

observe: $\forall \sigma \in S_n \curvearrowright (\mathbb{R}P^\infty)^{\times n}$.

$\sigma^*((\gamma^1)^{\times n}) \cong (\gamma^1)^{\times n}$. (isomorphic bundles)

$\Rightarrow f \circ \sigma$ is also a classifying map of $(\gamma^1)^{\times n}$

$\Rightarrow f \circ \sigma \simeq f$

$\Rightarrow \sigma^* \circ f^* = f^*$ on $H^\bullet$.

⇒ $S_n$ acts trivially on

$$\mathrm{im}(f^*) \le H^\bullet((\mathbb{RP}^\infty)^{\times n}; \mathbb{Z}/2\mathbb{Z}).$$

or $\mathrm{im}(f^*) \le H^\bullet((\mathbb{RP}^\infty)^{\times n}; \mathbb{Z}/2\mathbb{Z})^{S_n}$

Recall. $H^\bullet((\mathbb{RP}^\infty)^{\times n}) = H^\bullet(\mathbb{RP}^\infty)^{\otimes n}$

$$= \mathbb{Z}/_2[a_1] \otimes \mathbb{Z}/_2[a_2] \cdots \otimes \mathbb{Z}/_2[a_n]$$

$$= \mathbb{Z}/_2[a_1, \dots, a_n].$$

Fundamental Thm of Symmetric Polynomials:

$$\Rightarrow \mathbb{Z}/_2[a_1, \dots, a_n]^{S_n} = \mathbb{Z}/_2[s_1, \dots, s_n].$$

Thus, $H^\bullet(G_n) \xrightarrow{f^*} H^\bullet(\mathbb{RP}^{\times n})^{S_n} \cong \mathbb{Z}/_2[s_1, \dots, s_n]$.

Claim: $w_k(\gamma^n) \longmapsto w_k((\gamma^1)^{\times n}) = s_k(a_1, \dots, a_n)$.

Hence, $\mathrm{Im} f^* = H^\bullet((\mathbb{RP}^\infty)^{\times n})^{S_n}$

[back to where skipped]

It suffices to prove that $f^*$ is injective.

We will show that $\forall r$.

$$\dim H^r(G_n) \leq \dim f^* H^r(G_n).$$

LHS $= \dim H^r(G_n) \leq$ # of $r$-cells in $G_n$,

$= \# \{ e(\sigma) : \sigma = (\sigma_1, \dots, \sigma_n)$ is a seq. $\sigma_1 < \sigma_2 < \cdots < \sigma_n$ s.t.

$$\dim e(\sigma) = \sum_{i=1}^{n} \underbrace{(\sigma_i - i)}_{:= \mu_i \geq 0} = r . \} .$$

$= \# \{ (\mu_1, \dots, \mu_n) : 0 \leq \mu_1 \leq \cdots \leq \mu_n$ s.t. $\sum_{i=1}^{n} \mu_i = r \}$

" partitions of $r$ into a sum of at most $n$ positive integers."

$= \# \{ (r_1, \dots, r_n) : \sum_{i=1}^{n} i r_i = r , \ r_i \geq 0 \ \forall i \}.$

via $(r_1 \cdots r_n) \Leftrightarrow r_n \leq r_n + r_{n-1} \leq \cdots \leq r_n + r_{n-1} \cdots + r_1$

$=$ # of monomials $s_1^{r_1} \cdots s_n^{r_n}$
of total degree $\sum i r_i = r$
in $\mathbb{Z}/2\mathbb{Z}[s_1 \cdots s_n]$

note: each monomial $\in f^* H^r G_n$.

$\leq \dim f^* H^r G_n = \text{RHS}.$ $\square$.

Rmk: Note that once we proved that $f^*$ is an iso, all the "$\leq$" in the proof are actually "$=$". Thus.

(1) each Schubert cell $e(\sigma)$ represents a nontrivial element in $H_*(G_n ; \mathbb{Z}/2\mathbb{Z})$

(2) $\dim H^*(G_n ; \mathbb{Z}/2\mathbb{Z})$ = # of partitions of $r$ into $\leq n$ positive integers.

(3) all characteristic classes of $\mathbb{R}^n$-bundles are products of SW classes.

(4) SW classes are unique.

(pf of 4): Suppose $w$, $\tilde{w}$ are two theories satisfying the same 4 axioms.

Then as before

$$w((\gamma^1)^{\times n}) = \tilde{w}((\gamma^1)^{\times n})$$

$$= (1+a_1)\cdots(1+a_n)$$

Consider $f^*: H^\bullet(G_n) \xrightarrow{\cong} H^\bullet((\mathbb{RP}^\infty)^{\times n})^{S_n}$

$$w_k(\gamma^n) \longmapsto s_k(a_1,\dots,a_n)$$

$$\tilde{w}_k(\gamma^n) \longmapsto s_k(a_1,\dots,a_n)$$

$\Rightarrow$ $w_k(\gamma^n) = \tilde{w}_k(\gamma^n)$.

$\Rightarrow$ $w_k(\xi) = \tilde{w}_k(\xi)$ for any bundle $\xi$

$\square$.

$\boxed{\text{STOP}}$

# Construction of SW classes

**Thm**: SW classes exist.

i.e. There exists $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z})$ $\forall i$ satisfying the 4 axioms.

There are many ways to construct $w_i$'s.

① by directly computing (using spectral. seq.).

$$H^*(G_n ; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[s_1, \dots, s_n]$$

define $w_i(\xi) := f_\xi^* s_i$ , $B(\xi) \xrightarrow{f_\xi} G_n$

② via obstruction theory.

③ via Steenrod operations (MS §8,9,10)

④ via Leray-Hirsch theorem. (today).

<u>Reference</u>: Hatcher VBKT.

Randall Williams

Recall:

**Thm** (Leray-Hirsch).

Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fiber bundle s.t. for some commutative ring $R$.

(a) $H^n(F;R)$ is a finitely generated free $R$-module $\forall n$

(b) $\exists\, c_j \in H^{k_j}(E;R)$ s.t. $\{i^* c_j\}_j$ form an $R$-basis for $H^\bullet(F;R)$ for each fiber $F$.

(i.e. $H^\bullet(F;R) \cong R\{c_1, c_2, \dots\}$. ).

Then
$$H^\bullet(B;R) \otimes H^\bullet(F;R) \longrightarrow H^\bullet(E;R)$$
$$\sum_{i,j} b_i \otimes i^*(c_j) \longmapsto \sum_{i,j} p^*(b_i) \cup c_j$$
is an isomorphism of $R$-modules.

Or equivalently, $H^\bullet(E;R)$ is a free module over the ring $H^\bullet(B;R)$ with basis $\{c_j\}$.

$\boxed{\text{STOP}}$.

## Construction of SW classes

It suffices to consider when the base is a CW complex.

(if not, take a CW approximation

$$\begin{tikzcd} & \xi' \arrow[r] \arrow[d] & \xi \arrow[d] \\ CW. \arrow[r, squiggly] & B' \arrow[r] & B \end{tikzcd}$$

)

Given a vector bundle $\xi$:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

CW complex.

consider the projective bundle $P(\xi)$:

$$\begin{tikzcd} P(\mathbb{R}^n) = \mathbb{R}P^{n-1} \arrow[r] & P(E) \arrow[d, "P(\pi)"] \\ & B \end{tikzcd}$$

Fiber of $P(\xi)$ at $b \in B$

$= \{ L : L \text{ is a line in the fiber of } \xi \text{ at } b \}$

$\cong \mathbb{R}P^{n-1}$.

$P(\xi)$ is a fiber bundle.

**Goal**: Apply Leray-Hirsch to $P(\xi)$. $R = \mathbb{Z}/2\mathbb{Z}$.

Check the conditions:

(1) $H^i(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z}) = \langle a^i \rangle$ $\quad a \in H^1$. $\quad i \le n-1$. ✓

(2) want classes

$$\begin{tikzcd}
H^i(P(E);\mathbb{Z}/2\mathbb{Z}) \arrow[r, "i^*"] & H^i(\mathbb{RP}^{n-1};\mathbb{Z}/2\mathbb{Z}) \\
x_i \arrow[r, mapsto] & a^i
\end{tikzcd}$$

Recall last week:

we constructed a map $\hat{f} : \overset{E}{\overset{\shortparallel}{E(\xi)}} \to \mathbb{R}^\infty$ s.t. $\hat{f}$ is linear and injective on each fiber of $\xi$.

so $\mathbb{R}^n \to E \xrightarrow{\hat{f}} \mathbb{R}^\infty$ is injective.

$\Rightarrow$ we can projectivize:

$$P(\mathbb{R}^n) = \mathbb{RP}^{n-1} \xrightarrow{i} P(E) \xrightarrow{P(\hat{f})} \mathbb{RP}^\infty$$

Say $H^\bullet(\mathbb{R}P^\infty) = \mathbb{Z}/2\mathbb{Z}[\alpha]$, $\alpha \in H^1$.

Define $x := P(\hat{f})^*\alpha \in H^1(P(E))$

Then

$$\begin{tikzcd} H^\bullet(\mathbb{R}P^{n-1}) & H^\bullet(P(E)) \arrow[l, "i^*"'] & H^\bullet(\mathbb{R}P^\infty) \arrow[l, "P(\hat{f})^*"'] \\ \alpha^i & x^i \arrow[l, mapsto] & \alpha^i \arrow[l, mapsto] \end{tikzcd}$$

$$x^i \overset{!!}{=} x_i$$

(2) ✓

Apply Leray-Hirsch to

$$\begin{tikzcd} \mathbb{R}P^{n-1} \arrow[r, "i"] & P(E) \arrow[d, "P(\pi)"] \\ & B \end{tikzcd}$$

$$\Rightarrow \underset{\substack{\uparrow \\ \text{module}}}{H^\bullet(P(E))} = \underset{\substack{\uparrow \\ \text{ring}}}{H^\bullet(B)} \underset{\substack{\uparrow \\ \text{basis}}}{\{1, x, x^2, \dots, x^{n-1}\}}.$$

$\Rightarrow \exists!$ classes $w_1, w_2, \dots, w_n \in H^\bullet(B)$

s.t. $$x^n = \sum_{i=1}^{n} w_i \cdot x^{n-i}$$

Define SW classes:

$w_i(\xi) := w_i$ for $i = 1 \dots n$.

$w_0(\xi) := 1$. $w_i(\xi) = 0$ $\forall i > n$.

Or LS $\Rightarrow$ $H^\bullet(P(\xi)) \cong H^\bullet(B) \otimes H^\bullet(\mathbb{RP}^{n-1})$

$$\sum_{i,j} P(\pi)^*(b_i) \cup x^j \longmapsfrom \sum_{i,j} b_i \otimes a^j$$

in particular, $\exists!$ classes $w_i \in H^i(B)$, $i=1,\dots n-1$

s.t. $\underset{\substack{\cap \\ H^n(P(\xi))}}{x^n} = P(\pi)^*(w_1) \cup x^{n-1} + P(\pi)^*(w_2) \cup x^{n-2} + \cdots + P(\pi)^*(w_n) \cup x^0$

Or more briefly, $x^n + w_1 x^{n-1} + w_2 x^{n-2} + \cdots + w_n \cdot 1 = 0$

$\uparrow$

"projective bundle formula"

# Check the axioms

**A1:** $w_i(\xi) \in H^i(B(\xi); \mathbb{Z}/2\mathbb{Z})$.

$w_0(\xi) = 1$

$w_i(\xi) = 0 \quad \forall\, i > n.$

(by def.).

**A2** (Naturality): Suppose:

$$\begin{tikzcd} E' \arrow[r, "\tilde{g}"] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r, "g"] & B \end{tikzcd}$$

Consider

$$\begin{tikzcd} E' \arrow[r, "\tilde{g}"] \arrow[rr, bend right, "\hat{f}\circ\tilde{g}"'] & E \arrow[r, "\hat{f}"] & \mathbb{R}^\infty \end{tikzcd}$$

linear injection on fibers of $E'$.

$$\begin{tikzcd} H^\bullet(\mathbb{RP}^{n-1}) & H^\bullet(P(E')) \arrow[l] & H^\bullet(P(E)) \arrow[l, "P(\tilde{g})^*"'] & H^\bullet(\mathbb{RP}^\infty) \arrow[l] \\ a & x' = x(E') \arrow[l, mapsto] & x = x(E) \arrow[l, mapsto] & \alpha \arrow[l, mapsto] \end{tikzcd}$$

**exercise**: finish the check.

## A3 : ( Whitney product formula).

$$\omega_K ( \xi \oplus \eta) = \sum_i \omega_i (\xi) \cup \omega_{K-i} (\eta).$$

abbreviate : $E_1 := E(\xi)$ ,

$E_2 := E(\eta)$

$E_1 \oplus E_2 := E(\xi \oplus \eta)$.

$\dim(\xi) = m$. $\dim(\eta) = n$.

$i = 1, 2$, $E_i \hookrightarrow E_1 \oplus E_2 \longrightarrow \mathbb{R}^\infty$ linear injective on fibers.

$$\Rightarrow \quad P(E_i) \hookrightarrow P(E_1 \oplus E_2) \longrightarrow \mathbb{R}P^\infty$$

$$\Rightarrow \begin{tikzcd} H^*(P(E_i)) & H^*(P(E_1 \oplus E_2)) \arrow[l] & H^*(\mathbb{R}P^\infty) \arrow[l] \\ x(E_i) & x = x(E_1 \oplus E_2) \arrow[l, mapsto] & \alpha \arrow[l, mapsto] \\ 0 & u \arrow[l, mapsto] & \\ 0 & v \arrow[l, mapsto] & \end{tikzcd}$$

Define $u := \sum_j \omega_j (E_1) \, x^{m-j}$ , $u, v \in H^*(P(E_1 \oplus E_2))$

$v := \sum_j \omega_j (E_2) \, x^{n-j}$

claim: $uv = 0$ in $H^*(P(E_1 \oplus E_2))$

Claim $\Rightarrow$

$$0 = uv = \sum_j \Big( \underbrace{\sum_{r+s=j} w_r(E_1)\, w_s(E_2)}_{\substack{\text{must be } w_j(E_1 \oplus E_2) \\ \text{by definition.}}} \Big) x^{m+n-j}$$

pf of claim :

$$\begin{array}{lcl} P(E_1) & \subseteq & \\ & & P(E_1 \oplus E_2) \,. \qquad P(E_1) \cap P(E_2) = \emptyset \\ P(E_2) & \subseteq & \end{array}$$

$U_1 := P(E_1 \oplus E_2) \setminus P(E_1)$

$U_2 := \quad \cdots \quad \setminus P(E_2)$

$U_1$ deformation retracts onto $P(E_2)$.

$U_2$ $\quad$ - - - - - - $\quad$ $P(E_1)$

$$\begin{tikzcd}
H^m(P(E_1 \oplus E_2), P(E_1)) \arrow[r] \arrow[d, equal, "\wr"] & H^m(P(E_1 \oplus E_2)) \arrow[r] & H^m(P(E_1)) \arrow[r] & \cdots \\
\tilde{u} \arrow[r, mapsto] & u \arrow[r, mapsto] & 0 & \\
H^m(P(E_1 \oplus E_2), U_2) & & &
\end{tikzcd}$$

Same for $v$.

Now: $X = P(E_1 \oplus E_2)$.

$$\begin{tikzcd}
H^m(X, U_2) \times H^n(X, U_1) \arrow[rrr, "\cup"] \arrow[ddd] & & & H^{m+n}(X, U_1 \cup U_2) = 0 \arrow[ddd] \\
& (\tilde{u}, \tilde{v}) \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] & \\
& (u, v) \arrow[r, mapsto] & uv & \\
H^m(X) \times H^n(X) \arrow[rrr, "\cup"] & & & H^{m+n}(X)
\end{tikzcd}$$

$\square$

(A4): Consider $\gamma_1^1$ :

$$\begin{tikzcd}
\mathbb{R}^1 \arrow[r] & E \arrow[d, "\pi"] \\
& \mathbb{R}P^1
\end{tikzcd}$$

$$\begin{tikzcd}
P(\gamma_1^1) : \mathbb{R}P^0 = * \arrow[r] & P(E) \arrow[d, "P(\pi) = id"] \\
& \mathbb{R}P^1
\end{tikzcd}$$

Can define $\hat{f} : E \longrightarrow \mathbb{R}^\infty$ by

$$(\ell, v) \longmapsto v \in \mathbb{R}^2 \subseteq \mathbb{R}^\infty .$$

$P(\hat{f}) : P(E) = \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^\infty$

is the standard inclusion.

so
$$\begin{tikzcd}
H^1(P(E)) & H^1(\mathbb{R}P^\infty) \arrow[l, "P(\hat{f})^*"'] \\
x & \alpha \arrow[l, mapsto]
\end{tikzcd}$$

$x$ is a generator of $H^1(\mathbb{R}P^1)$.

Defining relation

$$x^1 = \omega_1 \cdot x^0 \quad \Rightarrow \quad \omega_1 = x \text{ is a generator}$$
for $H^1(\mathbb{RP}^1; \mathbb{Z}/2)$.

□.

## The splitting principle

**Thm**: For an n-plane bundle $\xi$ over a paracompact base $B$. There is a space $F(E)$ and a map $f: F(E) \to B$ s.t.

① $f^*\xi$ is a sum of line bundles over $F(E)$

② $f^*: H^{\circ}(B; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{\circ}(F(E); \mathbb{Z}/2\mathbb{Z})$ is injective.

**Pf**: By induction on n, it suffices to find $F(E) \xrightarrow{f} B$ s.t. $f^*\xi = \overset{\text{a line bundle}}{\eta} \oplus \eta^{\perp}$ and $f^*$ is injective.

Then iterate the process.

Now take $F(E) = P(E). \xrightarrow{f = P(\pi).} B.$

$$f^*: H^{\circ}(B) \longrightarrow H^{\circ}(P(E)) \text{ is injective.} ✓$$

$$\begin{tikzcd} H^{\circ}(P(E)) \arrow[d, phantom, "\shortparallel"] \\ H^{\circ}(B)\{1, x, \dots, x^{n-1}\}. \end{tikzcd}$$

Total space

$$E(f^*\xi) = \left\{ (b, L, v) \;\middle|\; b\in B,\ L \text{ is a line in } F_b\xi,\ v\in F_b\xi \right\}$$

a subbundle is:

$$E(\eta) = \left\{ (b. L, v) \;\middle|\; b\in B.\ L \text{ a line in } F_b\xi,\ v\in L \right\}$$

Hence, $f^*\xi = \eta \oplus \eta^\perp$ since $B$ is paracompact.

L7.

**Rmk:** If we iterate the process $n$ times,

$$F(E) = \left\{ (b, L_1, L_2, \dots, L_n) \;\middle|\; b\in B,\ L_1\oplus\cdots\oplus L_n = F_b\xi \right\}$$

$$\begin{tikzcd} GL_n(\mathbb{R})/T^n. \arrow[d, phantom, "\cong"] & & \\ \{\text{frames of } F_b\xi\} \arrow[r] & F(E) \arrow[r, "f"] & B \end{tikzcd}$$

## Application

**Goal**: Compute $w(\xi \otimes \eta)$ in terms of $w(\xi)$ and $w(\eta)$

**Step 1.** (HW).

Suppose $\xi, \xi'$ are sum of line bundles.

$\xi = L_1 \oplus \cdots \oplus L_n$

$\xi' = L_1' \oplus \cdots \oplus L_m'$

$$w(\xi \otimes \xi') = w(\bigoplus_{i,j} L_i \otimes L_j') = \prod_{i,j} w(L_i \otimes L_j')$$

**Step 2.**

Use splitting principle for general case.

Take $F \xrightarrow{f} B$ s.t.

$f^*\xi$ and $f^*\xi'$ both split as sums of line buns.

e.g. take $F_1 \xrightarrow{f_1} B$ s.t. $f_1^*\xi$ splits.

then take

$$\begin{tikzcd} F_2 \arrow[r, "f_2"] \arrow[rr, bend right, "f"'] & F_1 \arrow[r, "f_1"] & B \end{tikzcd}$$

s.t. $f_2^*(f_1^*\xi')$ splits.

Let $F_2 = F$. $f = f_1 \circ f_2$.

Then we have determined

$$\begin{tikzcd} w(f^*(\xi) \otimes f^*(\xi')) & \in & H^\bullet(F) \\ w(\xi \otimes \xi') \arrow[u, "f^*"] & \in & H^\bullet(B) \arrow[u, hook, "f^*"'] \end{tikzcd}$$

The same formula holds for

$$w(\xi \otimes \xi').$$

Today: Orientation & Euler class.

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) = \mathbb{Z}$$

Recall: $\nearrow$ rank n vect. sp..

An orientation of $V$, $\dim_{\mathbb{R}} V = n$.

|| a choice of component of $f \in Iso(\mathbb{R}^n, V) \cong GL_n(\mathbb{R})$.

|| a choice of generator

$$\begin{tikzcd} u_V \arrow[d, mapsto] & \in H^n(V, V_0; \mathbb{Z}) \arrow[d, "f^*"] \\ {} & \in H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; \mathbb{Z}) = \mathbb{Z} \end{tikzcd}$$

$V_0 := V \setminus \{0\}$.

|| a choice of

- a component of

$$\begin{tikzcd} \Lambda^n V \setminus 0 \arrow[r, "\cong"] & \Lambda^n \mathbb{R}^n \setminus 0 \arrow[d, "\det."', "IIS."] \\ {} & \mathbb{R} \setminus 0 \end{tikzcd}$$

$\xi$ = rank-$n$ vector bundle $/\mathbb{R}$.

__Def__: An __orientation for $\xi$__ is a nonvanishing section of $\Lambda^n \xi$

$$\begin{tikzcd} \xi: & F \arrow[r] & E \arrow[d] \\ & & B \end{tikzcd}$$

$$\begin{tikzcd} \Lambda^n \xi: & \mathbb{R} \cong \Lambda^n F \arrow[r] & E(\Lambda^n \xi) \arrow[d, shift right] \\ & & B \arrow[u, dotted, shift right, "s"'] \end{tikzcd}$$

An orientation gives a preferred generator $u_F \in H^n(F, F_0; \mathbb{Z})$ on each fiber $F$. that is locally compatible.

__Notation__: $E_0 = E \setminus \{\text{zero section}\}$

**Thm**: For $\xi$ an oriented rank n bundle,

there is a unique class

$$u \in H^n(E, E_0 ; \mathbb{Z}).$$

that restricts to the prefered generator

$$u_F \in H^n(F, F_0 : \mathbb{Z})$$

on each fiber $F$.

Rmk: Without an orientation on $\xi$, the statement holds for $R = \mathbb{Z}/2\mathbb{Z}$.

Rmk. Such $u \overset{= u(\xi)}{\in} H^n(E, E_0; \mathbb{Z})$ is called the Thom class of the oriented bundle $\xi$.

pf sketch. (Same as our construction of fundamental class in AT I)

prove the statement for

- trivial $\xi$
- $B = U_1 \cup U_2$ open trivialization of $\xi$
- $B$ = compact.
- $B = \bigcup$ compact subsets.

⋮

**Thm.** ( Thom Isomorphism Thm)

Suppose $\xi : F \to E \xrightarrow{p} B$ is oriented with Thom class $u \in H^n(E, E_0; \mathbb{Z})$.

Then 
$$\begin{aligned} \phi : H^i(B;\mathbb{Z}) &\longrightarrow H^{n+i}(E, E_0 ; \mathbb{Z}) \\ \alpha &\longmapsto p^*\alpha \cup u \end{aligned}$$
is an isomorphism $\forall i \geq 0$.

**Pf**: Apply Leray-Hirsch to the pair:

$$(F, F_0) \longrightarrow (E, E_0) \longrightarrow B.$$

$$\begin{tikzcd} H^n(F, F_0; \mathbb{Z}) & H^n(E, E_0; \mathbb{Z}) \arrow[l] \\ u|_F & u \arrow[l, mapsto] \end{tikzcd}$$

$\wr$

$H^i = 0 \quad \forall i \neq n.$

so $H^*(E, E_0) \cong H^*(B)\{u\}$

$\uparrow$ single basis element.

$\square$.

<u>Def</u>: The <u>Euler class</u> of an oriented $n$-plane bundle $\xi$ is

$e(\xi) \in H^n(B; \mathbb{Z})$

s.t.
$$\begin{tikzcd} H^n(B;\mathbb{Z}) \arrow[r, "p^*", "\cong"'] & H^n(E;\mathbb{Z}) & H^n(E, E_0; \mathbb{Z}) \arrow[l] \\ e(\xi) \arrow[rr, mapsto, bend left] \arrow[r, mapsto] & u(\xi)|_E & u(\xi) \arrow[l, mapsto] \end{tikzcd}$$

<u>properties</u>:

1. (Naturality) If $f: B \to B'$ is covered by an orientation preserving bundle map $\xi \to \xi'$, then $e(\xi) = f^* e(\xi')$.

2. Let $\bar{\xi} := \xi$ with opposite orientation

then $e(\bar{\xi}) = -e(\xi)$.

3.

$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta)$$

$$e(\xi \times \eta) = e(\xi) \times e(\eta).$$

4. If $\xi$ has a nonvanishing section

then $e(\xi) = 0$.

Pf. $\xi$ has section $\Rightarrow$ $\xi = \varepsilon \oplus \varepsilon^{\perp}$ (by a metric)

$$\Rightarrow \quad e(\xi) = \underset{\substack{\shortparallel \\ 0}}{e(\varepsilon)} \cup e(\varepsilon^{\perp}) = 0. \qquad \square$$

5.

**Thm:** If $M$ is smooth compact oriented then $e(\tau_M) = \chi(M) \cdot PD(a\ pt)$.

or. $\langle e(\tau_M), [M] \rangle = \chi(M)$.

**Thm:** If $\xi$ is an oriented vector bundle over a closed oriented manifold $B$

then $e(\xi) = PD(Z_s)$

where $Z_s = \{ b \in B \mid s(b) = 0 \}$ for $s$ a section transverse to the zero section.

6. If $\xi$ is orientable $\mathbb{R}^n$-bundle, then

$$\begin{array}{ccc} H^n(B;\mathbb{Z}) & \longrightarrow & H^n(B;\mathbb{Z}/2\mathbb{Z}) \\ e(\xi) & \longmapsto & w_n(\xi). \end{array}$$

# Intersection theory:

Suppose $X$ is a compact oriented $n$ manifold

$$PD_X : H_i(X\,{}^{,\partial X}) \longrightarrow H^{n-i}(X).$$

$$\alpha \cap [X] \longleftarrow\!\shortmid\ \alpha \qquad \cap \text{ "cap product"}$$

Intersection product:

$$H_i(X) \times H_j(X) \xrightarrow{\bullet} H_{n-i-j}(X)$$

$$(a,b) \longmapsto a \bullet b$$

s.t. $PD_X(a\bullet b) = PD_X(b) \cup PD_X(a)$

__Thm__: $A, B \subseteq X$ oriented submanifolds s.t. $A \pitchfork B$.

Then $[A]_X \cdot [B]_X = [A\cap B]_X$.

where $[A]_X := i_*[A]$. $A \xhookrightarrow{i} X$.

__Rmk__: Similar statement holds if $\partial X = \phi$

we replace $H^\circ(X)$ by $H^\circ(X, \partial X)$

and require $A, B$ to be transverse to $\partial X$.

__pic__: $\pitchfork$ intersection $\xleftrightarrow{PD}$ cup product.

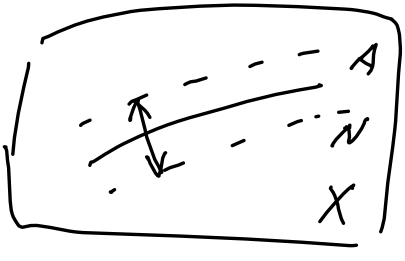
Given $A \subseteq X$. $(A \pitchfork \partial X)$.

define $\tau_A^X := PD_X(\, i_*[A]\,) \in H^{x-a}(X, \partial X)$

or $i_*[A] = \tau_A^X \cap [X]$. $\quad i : A \hookrightarrow X$.

Tubular neighborhood thm $\Longrightarrow$

$E(\nu_A^X) \cong N$ a tubular nbhd.



Then

$$\begin{tikzcd} H^{x-a}(E(\nu_A^X), E_0(\nu_A^X)) \arrow[r, "\cong"] & H^{x-a}(N, N \setminus A) \arrow[r, "\cong", "\text{excision}"'] & H^{x-a}(X, X \setminus N) \arrow[d] \\ u(\nu_A^X) \arrow[rr, mapsto] & & \tau_A^X \in H^{x-a}(X) \end{tikzcd}$$

Thom class of normal bun

"Thom class of submanifold"

pf sketch:

key: $A \pitchfork B \Rightarrow \nu^{A}_{B\cap A} \cong \nu^{X}_{B}\big|_{B\cap A}$

$$B\cap A \xhookrightarrow{i} B$$

check on fibers: $A \pitchfork B \Rightarrow T_pA + T_pB = T_pX$ )

$$\Rightarrow \frac{T_pA}{T_p(B\cap A)} \xrightarrow[\cong]{T(i^X_A)} \frac{T_pX}{T_pB}$$

Hence, $\boxed{\tau^{A}_{B\cap A} = (i^X_A)^* \, \tau^X_B .}$ (☆)

Now:

$$[B\cap A]_X \overset{\text{def}}{=} (i^X_{B\cap A})_* \, [B\cap A]$$

$$= (i^X_A)_* \underbrace{(i^A_{B\cap A})_* \, [B\cap A]}_{''\ \tau^A_{B\cap A} \cap [A]}$$

$$= (i^X_A)_* \left(\tau^A_{B\cap A} \cap [A]\right)$$

$$= (i_A^X)_* \left( (i_A^X)^* \tau_B^X \cap [A] \right) \quad \text{by } (\star).$$

$$= \tau_B^X \cap \left[ (i_A^X)_* [A] \right]$$

in general, $i_* ( i^* a \cap b) = a \cap i_* b.$

$$\Big\downarrow =$$

$$= \tau_B^X \cap \left( \tau_A^X \cap [X] \right)$$

$$= \left( \tau_B^X \cup \tau_A^X \right) \cap [X]$$

$$= PD_X^{-1} \left( \tau_B^X \cup \tau_A^X \right)$$

$$= PD_X^{-1} \left( PD_X([B]_X) \cup PD_X([A]_X) \right).$$

$$= [B]_X \cdot [A]_X$$

$\square$

Recall last time:

$\xi$ oriented $n$-dim

$$\begin{tikzcd}
H^n(E, E_0; \mathbb{Z}) \arrow[r] & H^n(E) & H^n(B) \arrow[l, "\cong"{above}, "\pi^*"{below}] \\
u(\xi) \arrow[rr, mapsto] & & e(\xi) \\
\text{Thom class} & & \text{Euler class}
\end{tikzcd}$$

<u>Thm</u>: If $\xi$ is an oriented vector bundle over a closed oriented manifold $B$ then $e(\xi) = PD_B([Z_s])$ where $Z_s = \{ b \in B \mid s(b) = 0 \}$ for $s$ a section transverse to the zero section.

Recall Thm.

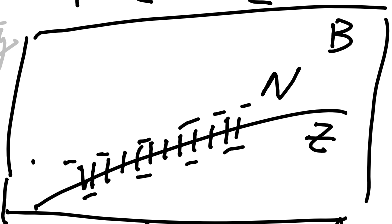
**pf sketch**: Let $Z := Z_s \subseteq B$

Tubular neighborhood thm

$$\Rightarrow \quad E(\nu_Z^B) \cong N \subseteq B$$

zero section of $\nu_Z^B \cong \overset{\cup\!|}{Z}$

use transversality



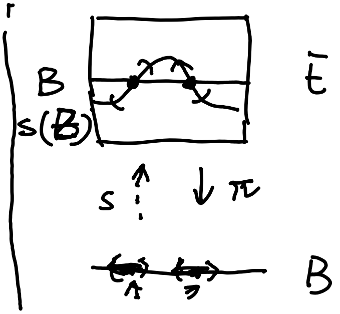
say $n = \dim B - \dim Z = \dim(\nu_Z^B)$

Back to the original bundle $\xi : E \to B$.

**Note**: $\nu_Z^B \cong \xi|_Z$ $\quad \boxed{Z \hookrightarrow B \xrightarrow{s} E}$

$$H^n(E, E_0) \xrightarrow{s^*} H^n(N, N\setminus Z) \to H^n(B, B\setminus Z)$$

$$u(\xi) \mapsto u(\nu_Z^B) = \tau_Z^N \mapsto \underset{\displaystyle PD_B[Z].}{\underset{\|}{\tau_Z^B}}$$



$$\begin{tikzcd}
H^n(E, E_0) \arrow[r, "s^*"] \arrow[d] & H^n(N, N\setminus Z) \arrow[d, "\cong", "\text{excision}"'] \\
H^n(E) \arrow[d, shift right, "s^*"'] & H^n(B, B\setminus Z) \arrow[d] \\
H^n(B) \arrow[u, shift right, "\pi^*"'] \arrow[r, equal] & H^n(B)
\end{tikzcd}$$

$$\begin{tikzcd}
u(\xi) \arrow[r, mapsto] \arrow[d, mapsto] & \tau_Z^N \arrow[d, mapsto] \\
e(\xi) \arrow[r, phantom, "="] & PD_B[Z]
\end{tikzcd}$$

□

**Thm**: $B = M^n$ closed oriented manifold

then

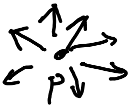
$$\langle \underset{\underset{H^n}{\cap}}{e(T_M)}, \underset{\underset{H_n}{\cap}}{[M]} \rangle = \chi(M).$$

(This is equivalent to Poincaré-Hopf.)

**pf sketch**: $\langle e(T_M), [M] \rangle$

$$= \langle PD_M([Z_s]), [M] \rangle$$

$$= \langle \underbrace{PD_M^{-1}[M]}_{\overset{\shortparallel}{1 \in H^0(M;\mathbb{Z})}}, Z_s \rangle$$



$$= \sum_{p \in Z_s} \underbrace{\text{Index of } s \text{ at } p}_{\shortparallel}.$$

sign of $\underset{\overset{\neq}{0} \text{ by } \pitchfork.}{\det}(\nabla s : T_pM \to T_pM)$

Poincaré-Hopf

$= \quad \chi(M)$

$\square$

**Example**: $\tau_{S^n}$ is nontrivial $\forall n$ even.

**note**: $\langle e(\tau_{S^n}), [S^n]\rangle = \chi(S^n) = 1+(-1)^n$

$= 2$ if $n$ even.

**Compare**: $w(\tau_{S^n}) = 1 \quad \forall n.$

$e(\tau_{S^n}) \bmod 2 = 0 = w_n(\tau_{S^n})$

**Applications**: (Intersection theory).

$$[A]\cdot[B]=[A\cap B].$$

(1) Cup product in $H^*(\mathbb{C}P^n;\mathbb{Z})$.

known $H^{2i}(\mathbb{C}P^n;\mathbb{Z})\cong\mathbb{Z}$ $\forall i\le n$

pick $h_i := \{[x_0:\cdots:x_n] \,/\, x_i=0\} \cong \mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

$[h_i] \in {}_{2n-2}(\mathbb{C}P^n;\mathbb{Z})$

**Intersection product •:**

$$\underset{H_{2n-2}}{\underset{\in}{[h_1]}}\cdot[h_2]\cdots[h_n] = [h_1\cap\cdots\cap h_n]$$

$$= \underset{[1:0:0\cdots:0]}{\underset{\|}{*}} \in H_0(\mathbb{C}P^n;\mathbb{Z})$$

Let $\tau_i := PD[h_i] \in H^2(\mathbb{C}P^n)$

or $\tau_1\cup\cdots\cup\tau_n = PD(*)$

**note**: $\tau_1=\tau_2=\cdots=\tau_n$ $\qquad \tau = PD(\underset{\text{hyperplane}}{\underset{\downarrow}{h}})$

so $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\tau]/\tau^{n+1}=0$.

## Application 2 : Bezout 's thm .

Suppose $F_1(x,y,z)$ , $F_2(x,y,z)$ are homogeneous polynomials (e.g. $x^3+y^3+z^3+xyz$ ).

For $i=1,2$, say $\deg F_i = d_i$ .

$$V_i := \{ [x:y:z] \in \mathbb{C}P^2 : F_i(x,y,z)=0 \}$$

" algebraic curve of degree $d_i$ "

**Thm.** (Bezout's thm).

If $V_1 \pitchfork V_2$,

then $[V_1 \cap V_2]_{\mathbb{C}P^2} = d_1 \cdot d_2 \cdot [*]_{\mathbb{C}P^2}$.

(generically, $V_1 \cap V_2$ at $d_1 d_2$ points).

say: "each point must have the same orientation since $V_1, V_2$ are complex submanifolds"

(everything holomorphic has a canonical orientation).

Pf: Let $\tau$ be the hyperplane class of $\mathbb{C}P^2$.

pick a generic hyperplane $H_1$ s.t. $H_1 \pitchfork V_1$.

Then $H_1 \cap V_1 = d_1$ points.

---

skip.

$H_1 \cap V_1 = \{ [x:y:z] \mid F_1(x,y,z) = 0$ and $ax+by+cz=0 \}$

say $c \neq 0$.

$= \{ [x : y : \overset{-\frac{ax+by}{c}}{\cancel{z}}] \mid F_1(x, y, -\frac{ax+by}{c}) = 0 \}$.

(say $x \neq 0$)

$= \{ [1 : y : -\frac{a+by}{c}] \mid F_1(1, y, -\frac{a+by}{c}) = 0. \}$.

if $x = 0$.

$( F_1(0, y, \frac{by}{c}) = \lambda \cdot y^{d_1}$ $\quad d_1$ points ,

:

$F_1(1, y, \frac{a+by}{c})$ is a nonhomogeneous poly. in $y$ with deg $d_1$ and has $d_1$ solutions generically.

$$\Rightarrow [H_1 \cap V_1]_{\mathbb{C}P^2} = d_1 [*]_{\mathbb{C}P^2}$$

$$\Rightarrow \tau \cup \underbrace{\tau(V_1)}_{PD[V_1]} = d_1 \cdot PD(*) .$$

$\tau(V_1) \in H^2(\mathbb{C}P^2) = \langle \tau \rangle .$

$\Rightarrow \exists\ m_1$ s.t. $\tau(V_1) = m_1 \tau .$

$$\Rightarrow \tau \cup \tau(V_1) = \tau \cup (m_1 \tau) = m_1 \tau^2$$

$$= m_1 \cdot PD(*)$$

$$= d_1 \cdot PD(*)$$

$\Rightarrow m_1 = d_1$

$$\Rightarrow \tau(V_1) = d_1 \cdot \tau$$

Similarly, $\tau(V_2) = d_2 \tau .$

:

Thus,

$$\tau(V_1) \cup \tau(V_2) = d_1 \tau \cup d_2 \tau$$
$$= d_1 d_2 \tau^2$$
$$= d_1 d_2 \, PD(*)$$

or $$[V_1] \cdot [V_2] = d_1 d_2 [*]$$
$$\shortparallel$$
$$[V_1 \cap V_2]$$

$\square$

This argument can be easily generalised: (Bezout in n-dim).

<u>Thm</u>: Suppose $\forall i = 1, \cdots, n$,

$F_i(x_0, \cdots, x_n)$ is a homo. poly of deg $d_i$

and $V_i := \{ [x_0 \cdots x_n] \in \mathbb{C}P^n : F_i = 0 \}$.

and $\forall i \neq j$, $V_i \pitchfork V_j$.

Then $[V_1 \cap \cdots \cap V_n]_{\mathbb{C}P^n} = d_1 d_2 \cdots d_n \cdot [*]_{\mathbb{C}P^n}$.

# Complex vector bundles

$\pi : E \to B$

fibers are $\mathbb{C}$-vector spaces ($\cong \mathbb{C}^n$).

**Examples:** $\gamma_n^1$ over $\mathbb{C}P^n$.

$$\begin{tikzcd}
\mathrm{Isom}_{\mathbb{C}}(\mathbb{C}^n, V) \cong GL_n(\mathbb{C}) \arrow[d, hook, "\text{by picking an } \mathbb{R}\text{-iso } \mathbb{R}^{2n} \xrightarrow{\cong} \mathbb{C}^n"] & \text{1-component.} \\
\mathrm{Isom}_{\mathbb{R}}(\mathbb{R}^{2n}, V) \cong GL_{2n}(\mathbb{R}) & \text{2 components.}
\end{tikzcd}$$

Hence, if $\omega$ is a $\overset{n\text{-}\dim_{\mathbb{C}}}{\text{complex}}$ bundle

consider $\omega_{\mathbb{R}}$ as a $2n$-$\dim_{\mathbb{R}}$ real bundle

Then $\omega_{\mathbb{R}}$ is <u>oriented</u>.

Define $(\gamma_1^1)^\vee := \mathrm{Hom}_{\mathbb{C}}(\gamma_1^1, \varepsilon)$.

<u>Claim</u>: $e((\gamma_1^1)^\vee_{\mathbb{R}}) = \tau \in H^2(\mathbb{CP}^1; \mathbb{Z})$

Pf: Fix a vector $v \in \mathbb{C}^2$

we have a section of $\bar{\gamma}_1^1$:

$$\begin{tikzcd} \mathbb{CP}^1 \arrow[r, "s"] & E((\gamma_1^1)^\vee) \\ L \arrow[r, mapsto] & \left( \begin{array}{c} L \to \mathbb{C} \\ u \mapsto \langle u, v \rangle \end{array} \right) \in L^\vee \end{tikzcd}$$

$\to \mathbb{C}$-linear

$Z_s = \{[v]\}$ a point in $\mathbb{CP}^1$.

so $e(\bar{\gamma}_1^1) = PD([Z_s]) = PD([*]) = \tau$. $\square$

You check:

$$e((\gamma_n^1)^\vee) = \tau \in H^2(\mathbb{CP}^n; \mathbb{Z})$$

Suppose ω is a complex v.b.

$$\mathbb{C}^n \to E \xrightarrow{\pi} B.$$

$\mathbb{C}$-projectivize $\Downarrow$

$$\mathbb{C}P^{n-1} \to P(E) \to B$$

$\Downarrow$ Leray-Hirsch

key difference: $H^*(\mathbb{C}P^{n-1};\mathbb{Z}) = \mathbb{Z}[b]/(b^n=0)$.

Fix $b \in H^2(\mathbb{C}P^{n-1};\mathbb{Z}) \cong \mathbb{Z}$ a generator.

$b = -\tau$ $\tau =$ hyperplane class. so $\langle b, [\mathbb{C}P^1]\rangle = -1$.

[why sign? we want. ~~(later)~~

$$b = e(\gamma^1_{n-1}) = -\tau.$$

]

so $H^\bullet(P(E);\mathbb{Z}) = H^\bullet(B;\mathbb{Z})\{1, x, \dots, x^{n-1}\}$

where $H^\bullet(P(E);\mathbb{Z}) \longrightarrow H^\bullet(\mathbb{C}P^{n-1};\mathbb{Z})$

$$x \longmapsto b \in H^2$$

Define <u>Chern classes</u> of complex vector bundle $\omega$

to be $c_i := c_i(\omega) \in H^{2i}(B(\omega);\mathbb{Z})$

s.t.

$$(*) \cdots \quad x^n - c_1 x^{n-1} + c_2 x^{n-2} + \cdots (-1)^n c_n \cdot 1 = 0 .$$

in $H^\bullet(P(E);\mathbb{Z})$.

**Thm:** Chern classes satisfy the following:

(1) $c_0 = 1$, $c_i = 0 \quad \forall i > n = \dim_{\mathbb{C}}(\xi)$

(2) $c_i(f^*\xi) = f^* c_i(\xi)$

(3) $c(\xi \oplus \eta) = c(\xi) \cup c(\eta)$

where $c := \sum_{i=0}^{\infty} c_i$

(4) For the canonical line bundle $\gamma_1^1$

$$\begin{tikzcd} \mathbb{C} \arrow[r] & E = \{(v, L) \mid v \in L\} \arrow[d] \\ & \mathbb{CP}^1 = \{L \mid L \subseteq \mathbb{C}^2,\ \dim_{\mathbb{C}} L = 1\} \end{tikzcd}$$

$c_1(\gamma_1^1) = \overset{= -z}{b} \in H^2(\mathbb{CP}^1; \mathbb{Z})$ s.t. $\langle b, [\mathbb{CP}^1] \rangle = -1$

**Note:** The choice of sign in (4) is consistent with (*).

$P(E) = \mathbb{CP}^1$. $\quad x - c_1 = 0 \Rightarrow x = c_1 = b$.

## Sign convention :

We chose the signs on $c_i$ s.t.

$$c_1(\gamma_n^1) = e((\gamma_n^1)_{\mathbb{R}})$$

As a consequence, we have that for any complex n-plane bundle $\omega$

$$\boxed{c_n(\omega) = e(\omega_{\mathbb{R}})} \in H^{2n}(B;\mathbb{Z}).$$

( pf : similar as for SW classes via splitting principle ).

If we chose $b = +\tau \in H^2(\mathbb{C}P^{n-1})$ then we would have $e(\omega_{\mathbb{R}}) = (-1)^n c_n(\omega)$.

We can prove various similar properties of $c_i$ as $\omega_i$:

such as:

- $c(\varepsilon^n) = 1$
- $c_i(\xi \oplus \varepsilon^n) = c_i(\xi)$. $\forall i$.
- $\xi^n$ has $k$ independent sections
  $\Leftrightarrow$ $\xi^n = \varepsilon^k \oplus \eta^{n-k}$
  $\Leftrightarrow$ $c_i(\xi) = c_i(\eta^{n-k}) = 0$ $\forall i > n-k$.
- If $\xi \cong \eta$. then $c(\xi) = c(\eta)$.
- $\gamma_n^1$:
$$\begin{tikzcd} \mathbb{C} \arrow[r] & E \arrow[d] \\ & \mathbb{C}P^n. \end{tikzcd}$$
  Then $c(\gamma_n^1) = 1+b$
- Every $\omega$ over a paracompact base has a Hermitian inn. pro.
  $\langle w, v\rangle = \overline{\langle v, w\rangle} \in \mathbb{C}$.
- ~~$c(\tau_{\mathbb{C}P^n}) = (1+b)^{n+1}$~~

## Complex Grassmannian :

$$G_n(\mathbb{C}^{n+k}) := \{ X \mid X \subseteq \mathbb{C}^{n+k}.\ \dim_{\mathbb{C}} X = n \}$$

$$G_n(\mathbb{C}^{n+k}) \cong U_{n+k} \big/ U_n \times U_k$$ compact manifold

**Thm.** Every complex n-plane bundle over a paracompact base admits a bundle map into the canonical bundle $\gamma^n$ over $G_n(\mathbb{C}^\infty)$.

$$\begin{tikzcd} X \arrow[r] & E(\gamma^n) = \{ (X,v) \mid v \in X \}. \arrow[d] \\ & G_n(\mathbb{C}^\infty) = \{X\}. \end{tikzcd}$$

**pf:** Use a partition of unity exactly the same as $\mathbb{R}$ case.

Thm. $H^\bullet(G_n(\mathbb{C}^\infty);\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots, c_n]$.

pf 1: Similar to the real case. (you).

pf 2: Spectral sequence. (sketch):

Define $V_n(\mathbb{C}^{n+k}) :=$ "Stiefel man."

$$\{(v_1, \dots, v_n) \mid v_i \in \mathbb{C}^{n+k}, \text{ orthonormal}\}.$$

We have a fiber bundle

$$\begin{tikzcd}
U_n \arrow[r] & V_n(\mathbb{C}^{n+k}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & G_n(\mathbb{C}^{n+k}) \arrow[d, phantom, sloped, "\cong"] \\
U_n \arrow[r] & \dfrac{U_{n+k}}{1 \times U_k} \arrow[r] & \dfrac{U_{n+k}}{U_n \times U_k}
\end{tikzcd}$$

You check:

$V_n(\mathbb{C}^{n+k})$ is $2k$-connected

(or $U_k \hookrightarrow U_{n+k}$ induces iso on $\pi_{<2k}$).

so $V_n(\mathbb{C}^\infty) \simeq *$.

So we have a fibration

$$\begin{tikzcd} U_n \arrow[r] & V_n \arrow[r] \arrow[d, phantom, sloped, "\simeq"] & G_n . \\ & * & \end{tikzcd}$$

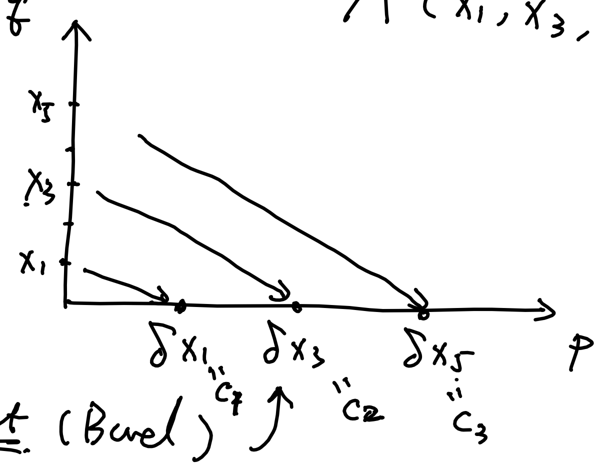
LSSS :

$$E_2 = H^*(G_n) \otimes H^*(U_n) \Rightarrow H^*(*)$$

<u>Recall</u>: ~~H~~

$$H^\bullet(U_n) \cong H^\bullet(S^{2n-1} \times S^{2n-3} \times \cdots \times S^1)$$

$$= \Lambda^\bullet(x_1, x_3, \dots, x_{2n-1})$$



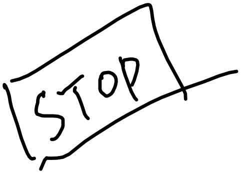
**Fact** (Borel)

$$H^\bullet(G_n) = \mathbb{Z}[c_1, c_2, \dots].$$

□.

__Say__: So far we see the theory of $\mathbb{C}$-vector bundles is similar to that of $\mathbb{R}$-vector bundles.

Next we will discuss their difference



Today:

vector bundles $/\mathbb{C}$ vs. vector bun $/\mathbb{R}$.

# (I) Complex conjugate bundle

A complex structure on a real 2n-plane bundle$_{\omega}$ $\xi$ is a cont. map

$$J: E(\xi) \to E(\xi) \qquad \mathbb{R}^{2n}$$

which restricts to an $\mathbb{R}$-linear map on fibers and $J(J(v)) = -v \quad \forall v \in E(\xi)$.

**Rmk:** ① real $\mathbb{R}^{2n}$-bundle $\xi$ + complex structure

$\Longleftrightarrow$ a complex $\mathbb{C}^n$-bundle

② Suppose $M^{2n}$ is a real manifold.

A complex str. on $TM$ is called an almost complex str. on $M$.

It turns out that

real manifold $M^{2n}$ + al. com. str. $\overset{J}{\vee}$ on $M$.

+ extra condition on $J$ $\Leftrightarrow$ $M$ a complex manifold.

Given a complex bundle $\xi$, its *conjugate bundle* $\bar{\xi}$ is the complex vector bundle s.t. $E(\xi) = E(\bar{\xi})$ with the identity map $E(\xi) \xrightarrow{f} E(\bar{\xi})$ complex conjugate linear.

i.e. $f(\lambda \underset{\substack{\uparrow \\ \xi}}{\cdot} e) = \bar{\lambda} \underset{\substack{\uparrow \\ \bar{\xi}}}{\cdot} f(e) \quad \forall \lambda \in \mathbb{C}$

$\forall\, e \in E(\xi) = E(\bar{\xi})$.

So $J_{\xi} = -J_{\bar{\xi}}$.

Suppose $\xi$ admits an Hermitian metric

$$\langle \lambda v, w \rangle = \lambda \langle v, w \rangle$$

$$\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle.$$

$$\langle v, w \rangle = \overline{\langle w, v \rangle}.$$

Then $\bar{\xi} \cong \mathrm{Hom}_{\mathbb{C}}(\xi, \overset{\text{trivial } \mathbb{C}\text{-bundle}}{\varepsilon})$

$v \longmapsto (u \longmapsto \langle u, v \rangle)$

However, in general $\xi \not\cong \bar{\xi}$.

(This is different from the real case).

<u>Example</u>: Consider $\gamma^1$:

$$\begin{tikzcd} \mathbb{C} \arrow[r] & E \arrow[d] \\ & \mathbb{CP}^\infty \end{tikzcd}$$

note: $\gamma^1 \otimes \overline{\gamma^1} \cong \gamma^1 \otimes \mathrm{Hom}(\gamma^1, \varepsilon) \cong \varepsilon$

so $c_1(\gamma^1) + c_1(\overline{\gamma^1}) = c_1(\varepsilon) = 0$

$c_1(\overline{\gamma^1}) = -c_1(\gamma^1) = \underset{\substack{\neq \\ b}}{-b} \in H^2(\mathbb{CP}^\infty; \mathbb{Z})$.

so $\gamma^1 \not\cong \overline{\gamma^1}$.

(In AG, $\gamma^1 = \mathcal{O}(-1)$, $\overline{\gamma^1} = \mathcal{O}(1)$).

<u>Rmk</u>: complex conjugation changes orientation on $\overset{\overset{\mathbb{R}^2}{''}}{\mathbb{C}}$

so $(\bar{\xi})_{\mathbb{R}} = \overline{(\xi_{\mathbb{R}})}$.

# (II) Connection btw SW and Chern classes.

Thm: Suppose $\xi$ is a complex vector bundle

then $$\omega_{2i+1}(\xi_{\mathbb{R}}) = 0$$

and $$\omega_{2i}(\xi_{\mathbb{R}}) = c_i(\xi) \mod 2$$

i.e. $$\begin{tikzcd} H^{2i}(B;\mathbb{Z}) \arrow[r] & H^{2i}(B;\mathbb{Z}/2\mathbb{Z}) \\ c_i(\xi) \arrow[r, mapsto] & \omega_{2i}(\xi_{\mathbb{R}}) \end{tikzcd}$$.

$$\boxed{\text{in particular, } w_{2n}(\xi_{\mathbb{R}}) = c_n(\xi) = e(\xi_{\mathbb{R}}) \mod 2}$$

Pf sketch: Say $\dim_{\mathbb{C}} \xi = n$.

note: $$\begin{tikzcd} \mathbb{RP}^1 \arrow[r] \arrow[d, phantom, "="] & \mathbb{RP}^{2n-1} \arrow[r, "p"] & \mathbb{CP}^{n-1} \\ \{L \mid L \subset \mathbb{C}\} \arrow[d, phantom, "\cong"] & L \arrow[r, mapsto] & L \otimes_{\mathbb{R}} \mathbb{C} \\ \mathbb{R}^2 & & \end{tikzcd}$$

consider $\hat{f} : E(\xi) \longrightarrow \mathbb{C}^{\infty}$

linear and injective on fiber.

We have:

$$\begin{tikzcd}
\mathbb{RP}^1 \arrow[d] & \mathbb{RP}^1 \arrow[d] & \mathbb{RP}^1 \arrow[d] \\
\mathbb{RP}^{2n-1} \arrow[r] \arrow[d] & \mathbb{RP}(E(\xi_{\mathbb{R}})) \arrow[r] \arrow[d] & \mathbb{RP}^\infty \arrow[d] \\
\mathbb{CP}^{n-1} \arrow[r] & \mathbb{CP}(E(\xi)) \arrow[r] & \mathbb{CP}^\infty
\end{tikzcd}$$

<u>note</u>:

$$\begin{tikzcd}
H^2(\mathbb{CP}^\infty;\mathbb{Z}) \arrow[r] & H^2(\mathbb{CP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[r] & H^2(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \\
\beta \arrow[r, mapsto] & \bar{\beta} \arrow[r, mapsto] & \alpha^2
\end{tikzcd}$$

$\beta \in H^2(\mathbb{CP}^\infty;\mathbb{Z})$
$\alpha \in H^1(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z})$ are standard generators.

[use LH].

$$\begin{tikzcd}
H^2(\mathbb{RP}(E);\mathbb{Z}/2\mathbb{Z}) & H^2(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[l] \\
x^2(\xi_{\mathbb{R}}) \arrow[r, leftrightarrow] & \alpha^2 \\
H^2(\mathbb{CP}(E);\mathbb{Z}/2\mathbb{Z}) \arrow[uu] & H^2(\mathbb{CP}^\infty;\mathbb{Z}/2\mathbb{Z}) \arrow[l] \arrow[uu] \\
\bar{x}_{\mathbb{C}}(\xi) \arrow[uu, mapsto] \arrow[r, leftrightarrow] & \bar{\beta} \arrow[uu, mapsto]
\end{tikzcd}$$

Chern classes mod 2 :

$$\overline{x}_{\mathbb{C}}(\xi)^{n} + \overline{c}_1(\xi)\,\overline{x}_{\mathbb{C}}(\xi)^{n-1} + \cdots + \overline{c}_n(\xi)\cdot 1 = 0$$

$$\in H^{2n}(\mathbb{C}P(E);\mathbb{Z}/2\mathbb{Z})$$

$$\big\downarrow$$

$$\left[x(\xi_{\mathbb{R}})^2\right]^{n} + \overline{c}_1(\xi)\left[x(\xi_{\mathbb{R}})^2\right]^{n-1} + \cdots = 0$$

$$\in H^{2n}(\mathbb{R}P(E);\mathbb{Z}/2\mathbb{Z})$$

$$\wr\!\!\downarrow$$

defining relations for $w_i(\xi_{\mathbb{R}})$'s.

$$\Rightarrow \quad w_{2i}(\xi_{\mathbb{R}}) = \overline{c}_i(\xi)$$

$$w_{2i+1}(\xi_{\mathbb{R}}) = 0 .$$

$\square$.

~~Say: (II) $\Rightarrow$ (I).~~

# Characteristic classes via obstruction theory

## (I) obstruction theory:

Q: When does a fiber bundle

$$\begin{tikzcd} F \arrow[r] & E \arrow[r] & B \arrow[l, dashed, bend right, "s"'] \end{tikzcd}$$

have a section?

Approach: For $B$ a CW complex.

try to build a section $s$

on each skeleton.

**Thm:** Suppose $F \to E \to B$ is a fiber bundle s.t. $F$ is $n$-simple, $B = CW$.

(i.e. $\pi_1(X, x_0) \curvearrowright \pi_n(X, x_0)$ trivially so $\pi_n(X, x_0) \cong \pi_n(X, y_0)$ $\cong$ is unique).

Let $s$ be a section on $B^{n-1}$ that can be extended to $B^n$. Then there is a unique and well-defined obstruction class $\longleftarrow$ local coefficients.

$$ob(s) \in H^{n+1}(B, \pi_n(F))$$

s.t. $ob(s) = 0$ iff $s$ can be extended to $B^{n+1}$.

pf idea :

**Rmk.** Finding a homotopy of sections $s \sim s'$

= find a section of

$$\begin{tikzcd} F \arrow[r] & E\times I \arrow[r] & B\times I \\ & & B\times\partial I \arrow[u, hook] \arrow[ul, dotted, "s\cup s'"'] \end{tikzcd}$$

extending $s, s'$

obstruction $\in H^{n+1}(B\times I, B\times\partial I ; \pi_n(F))$

$$\cong$$

$$H^n(B, \pi_n(F))$$

Suppose $F$ is $(n-1)$ - connected

i.e. $\pi_i(F) = 0 \quad \forall i \leq n-1$

obstruction to sections $\in H^i(B, \pi_{i-1}(F)) = 0 \quad \forall i \leq n$

...... homotopy of sections $\in H^i(B, \pi_i(F)) \quad \forall i \leq n-1$

$\exists$ s on $B^n$

and $s|_{B^{n-1}}$ is unique up to homotopy.

$\Rightarrow \exists !$ obstruction class ("first obstruction" or "primary obstruction")

$ob(s) \in H^{n+1}(B, \pi_n(F))$

independent of any choices.

STOP

## Additional remarks:

If $\pi_1 B \neq 0$.

① Local coefficient $\pi_1(B, b) \curvearrowright \pi_n(F_b)$

Take $\gamma \in \pi_1 B$. $\gamma : [0,1] \longrightarrow B$

$$\begin{tikzcd}
\gamma^* E \arrow[r, "\psi", "\cong"'] \arrow[d] & {[0,1] \times F} \\
{[0,1]} &
\end{tikzcd}$$

$$\begin{tikzcd}
F_b \arrow[r, "\psi|_0"] \arrow[rr, bend right, "\varphi_\gamma"'] & F & F_b \arrow[l, "\psi|_1"']
\end{tikzcd}$$

$\varphi_\gamma$ induces a ~~well-defined~~ map $\pi_n(F_b) \to \pi_n(F_b)$

if $F$ is $n$-simple. (base point),

only depend on $\gamma \in \pi_1 B$.

.l.F

③ first obstruction class is natural.

$$\begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

$F$ is $(n-1)$-connected

$$H^{n+1}(B, \pi_n(F)) \xrightarrow{f^*} H^{n+1}(B', \pi_n(F))$$

$$ob(\xi) \longmapsto ob(f^*\xi) = f^* ob(\xi)$$

# (I) Stiefel - Whitney classes as obstructions.

Define

$$V_k(\mathbb{R}^n) := \left\{ (v_1, \dots, v_k) \;\middle|\; \begin{array}{c}\text{orthonormal} \\ \text{in } \mathbb{R}^n\end{array} \right\}.$$

$$\cong O(n) / O(n-k). \quad \text{"Stiefel manifold"}.$$

Given $\xi : \mathbb{R}^n \to E \to B$.

consider $V_k(\xi) : V_k(\mathbb{R}^n) \to E \to B$.

$\xi$ has $k$ nowhere-dependent sections

$\Leftrightarrow$ $V_k(\xi)$ has a section.

Goal: SW classes

= first obstructions to a section of $V_k(\xi)$.

Preparation: Gysin sequence.

Thm: If $\xi$ is an oriented $n$-plane bundle, $\exists$ a LES:

$$\cdots \to H^i(B) \xrightarrow{\cup e} H^{i+n}(B) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \to H^{i+1}(B) \xrightarrow{\cup e} \cdots$$

where

$$\begin{tikzcd} \mathbb{R}^n \setminus 0 \arrow[r, hook] & E_0 := \{e \in E \mid e \neq 0\} = E \setminus i(B) \arrow[d, "\pi_0"] \\ & B \end{tikzcd}$$

$i :=$ zero section.

Pf: LES of pair $(E, E_0)$:

$$\begin{tikzcd} & \cdots \arrow[r] & H^j(E,E_0) \arrow[r] & H^j(E) \arrow[r] & H^j(E_0) \arrow[r, "\delta"] & H^{j+1}(E,E_0) \arrow[r] & \cdots \\ \text{Thom Iso.} \arrow[rr, hook] & & H^{j-n}(E) \arrow[u, "\cup u"', "\cong"] & & & & \\ & & H^{j-n}(B) \arrow[u, "\pi^*"', "\cong"] & & & & \\ & \cdots \arrow[r] & H^{j-n}(B) \arrow[r, "\cup e"] & H^j(B) \arrow[uuu, "\cong"] \arrow[r] & H^j(E_0) \arrow[r] & \cdots & \end{tikzcd}$$

Thom class:

$$u \in H^n(E,E_0) \to H^n(E) \xrightarrow{\cong} H^n(B)$$
$$u \longmapsto e$$

Rmk: If nonorientable same holds mod 2. □

Lemma: $\pi_i(V_k(\mathbb{R}^n)) = 0 \quad \forall i < n-k.$

$$\pi_{n-k}(V_k \mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } n-k \text{ even or } k=1 \\ \mathbb{Z}/2\mathbb{Z} & \text{else.} \end{cases}$$

Pf: Consider the fiber bundle

$$\begin{tikzcd} V_{k-1}(\mathbb{R}^{n-1}) \arrow[r] & V_k(\mathbb{R}^n) \arrow[r] & S^{n-1} \\ (v_1 \cdots v_{k-1}) \arrow[r, mapsto] & (v_1 \cdots v_k) \arrow[r, mapsto] & v_k \end{tikzcd}$$

Recall
$\pi_i S^n = 0$
$\forall i < n.$

iterate:

$$\begin{tikzcd} V_1(\mathbb{R}^{n-k+1}) \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & V_2(\mathbb{R}^{n-k+2}) \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & V_{k-1}(\mathbb{R}^{n-1}) \arrow[r, hook] \arrow[d] & V_k(\mathbb{R}^n) \arrow[d] \\ S^{n-k} & S^{n-k+1} & \cdots & S^{n-2} & S^{n-1} \end{tikzcd}$$

Apply $\pi_i$, we see $\pi_i(-) = 0$ for all spaces above if $i < n-k$. (LES of homotopy).

$\Rightarrow \pi_i(V_k(\mathbb{R}^n)) = 0$. if $i < n-k$.

$i = n-k$. consider $V_2(\mathbb{R}^{n-k+2})$, same $\pi_{n-k}$ as $V_k(\mathbb{R}^n)$

$$\begin{tikzcd}
S^{n-k} \arrow[r] \arrow[d, phantom, "\cap"] & V_2(\mathbb{R}^{n-k+2}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & S^{n-k+1} \arrow[d, phantom, sloped, "="] \\
TS^{n-k+1} = \mathbb{R}^{n-k+1} & UT(S^{n-k+1}) \arrow[r] \arrow[d, phantom, "\cap"] & S^{n-k+1} \\
& T(S^{n-k+1}) & \text{"unit tangent bundle"}
\end{tikzcd}$$

Gysin sequence :

$$\begin{tikzcd}
 & & \mathbb{Z} \arrow[r, "1 \mapsto \chi(S^{n-k+1})"] \arrow[d, phantom, sloped, "="] & \mathbb{Z} \arrow[d, phantom, sloped, "="] & \\
0 \arrow[r] & H^{n-k}(V_2) \arrow[r] & H^0(S^{n-k+1}) \arrow[r, "\cup e"] & H^{n-k+1}(S^{n-k+1}) \arrow[r] & \cdots
\end{tikzcd}$$

$$\to H^{n-k+1}(V_2) \to H^1(S^{n-k+1}) = 0.$$

Here, $e = e(TS^{n-k+1}) = \chi(S^{n-k+1}) \cdot PD(*)$.

if $n-k$ even. $e=0$.

$\Rightarrow H^{n-k}(V_2) = \mathbb{Z}$.

UCT + Hurewicz $\Rightarrow$ $\pi_{n-k}(V_2) = \mathbb{Z}$

if $n-k$ odd, $e = 2 \in \mathbb{Z} \cong H^{n-k+1}(S^{n-k+1})$

$\Rightarrow$ $H^{n-k}(V_2) = 0$, $H^{n-k+1}(V_2) = \mathbb{Z}/2\mathbb{Z}$

UCT + Hurewicz $\Rightarrow$ $\pi_{n-k}(V_2) = \mathbb{Z}/2$

□

Given $\xi$ an $\mathbb{R}^n$-bundle over $B$

$\forall k$, $V_k(\xi)$ is an $V_k(\mathbb{R}^n)$-bundle over $B$.

**Thm:** first obstruction class ($j := n-k+1$)

$$\begin{tikzcd}
o_j(\xi) \arrow[d, mapsto] & \in H^j(B, & \pi_{j-1}(V_{n-j+1}(\mathbb{R}^n))) \arrow[d] \\
w_j(\xi) & \in H^j(B, & \mathbb{Z}/2\mathbb{Z})
\end{tikzcd}$$

$\forall\ j \leq n$.

**Recall:** We proved before that

$$w_j(\xi) \neq 0 \Rightarrow V_{n-j+1}(\xi) \text{ has no section.}$$

Converse is not true sine

① When $j$ is even.

$$o_j(\xi) \in H^j(B, \tilde{\mathbb{Z}}) \text{ might be nonzero and even.}$$

② Even if $j$ is odd, $o_j(\xi) = w_j(\xi)$

There might be other obstruction after the first.

~~**Pf**: Consider universal bundle $\gamma^n$ over $G_n$~~

~~$H^*(G_n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_1 \cdots w_n]$~~

~~$\cup$~~

~~$H^j \ni o_j(\gamma^n) = f_j(w_1, \dots, w_n)$. $w_i = w_i(\gamma^n)$.~~

~~$= f(w_1, \dots, w_{j-1}) + \lambda w_j$ $\lambda \in \mathbb{Z}/2$.~~

~~this formula holds for any n-plane bundle.~~

~~Consider $\eta = \gamma^{j-1} \oplus \varepsilon^{n-j+1}$ over $G_{j-1}$.~~

~~$\eta$ has $n-j+1$ independent sections~~

~~$\Rightarrow$~~

~~$0 = o_j(\eta) \in H^j(G_{j-1}, \pi_{j-1} V_{n-j+1}(\mathbb{R}^n))$~~

~~(mod 2): $o_j(\eta) = f(w_1 \dots w_{j-1}) + \lambda w_j = 0$~~

~~note here $w_i = w_i(\eta^n) = w_i(\gamma^{j-1})$~~

~~$\Rightarrow f(w_1(\gamma^{j-1}), \dots, w_{j-1}(\gamma^{j-1})) = 0$ in $H^*(G_{j-1})$~~

~~but $w_i$'s are independent~~

~~$\Rightarrow f = 0$ polynomial.~~

Finally, need to show $\lambda \neq 0$. or $o_j(\gamma^n) \neq 0$.

Consider $j = n$.

$$\begin{tikzcd} & \gamma_1^n \arrow[r] \arrow[d] & \gamma^n \arrow[d] \\ \mathbb{R}P^n = G_1(\mathbb{R}^{n+1}) \arrow[r, phantom, "\cong"] & G_n(\mathbb{R}^{n+1}) \arrow[r] & G_n(\mathbb{R}^\infty) \end{tikzcd}$$

$H^\perp \leftarrow\!\shortmid H$

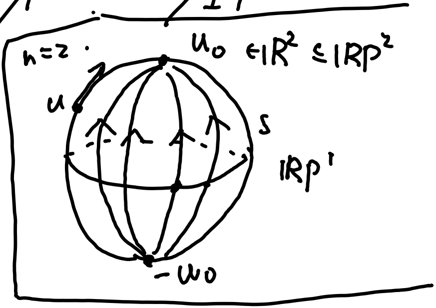
<u>Goal</u>: $O_n(\gamma_1^n) \neq 0$ in $H^n(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$

Fiber of $\gamma_1^n$ over $\{\pm u\} \in \mathbb{R}P^n = S^n / \pm 1$

is $u^\perp = \{ v \in \mathbb{R}^{n+1} \mid u \cdot v = 0 \}$

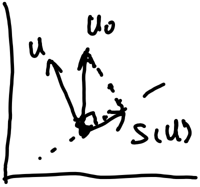
Fix $\pm u_0 \in \mathbb{R}P^n$

Define a section of $\gamma_1^n$ by



$$\{\pm u\} \xmapsto{s} u_0 - (u_0 \cdot u) u \quad (\perp u)$$

$s \neq 0$ unless at $\pm u_0$



Cell structure of $\mathbb{R}P^n$ :

$$\mathbb{R}P^n = \underset{\operatorname{int}(e^n)}{\mathbb{R}^n} \cup \mathbb{R}P^{n-1}$$

$\uparrow$ $(n-1)$-skeleton.

pick $\pm u_0 \in \mathbb{R}^n \subseteq \mathbb{R}P^n$;

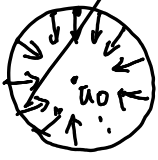
$s$ is a section on $(n-1)$-skeleton

obstruction cocycle:

$$o_n(\gamma_1^n) : C_n(\mathbb{R}P^n) \cong \mathbb{Z} \longrightarrow \pi_{n-1}(V_1 \mathbb{R}^n) \cong \mathbb{Z}$$

$\pi_{n-1}(V_1\mathbb{R}^n)$: $V_1\mathbb{R}^n = S^{n-1}$

$$[e^n] \longmapsto [\partial D^n = S^{n-1} \xrightarrow{-id} S^{n-1}]$$



For $j<n$. consider

$$\eta^n = \gamma_1^j \oplus \varepsilon^{n-j} \quad \text{over} \quad G_j(\mathbb{R}^{j+1}) \cong \mathbb{R}P^j$$

$$o_j(\eta) \in H^j(\mathbb{R}P^j, \pi_{j-1}(V_{n-j+1}(\mathbb{R}^n)))$$

$$\cong$$

$$\pi_{j-1}(V_1(\mathbb{R}^{j-1})) \quad (\text{mod } 2).$$

proceed with same argument.

$\boxed{\text{STOP}}$

## (II) Chern classes as obstruction.

$$V_{n-q}(\mathbb{C}^n) := \{ (n-q)\text{-frames in } \mathbb{C}^n \}.$$

(HW):

$$\pi_i V_{n-q}(\mathbb{C}^n) = \begin{cases} 0 & \forall i \leq 2q \\ \mathbb{Z} & i = 2q+1. \end{cases}$$

Given a complex $n$-plane bundle $\xi$.

first obstruction to a section of $V_{n-q}(\xi)$

$$\begin{tikzcd} o_{q+1}(\xi) \arrow[d] & \in & H^{2q+2}(B, \pi_{2q+1}(V_{n-q}(\mathbb{C}^n))) \arrow[d, phantom, "\wr"] \\ c_{q+1}. & & H^{2q+2}(B, \mathbb{Z}) \end{tikzcd}$$

(HW).

Recall: $c_{q+1} \neq 0 \Rightarrow \xi$ has no $(n-q)$ independent sections.

Converse might fail since there might be obstructions beyond the first.

## (III) Euler class as obstructions.

$\xi$ an $\mathbb{R}^n$-bundle.

consider $\overset{V_1(\xi)}{\overset{\shortparallel}{S(\xi)}}$ : $S^{n-1} \to S(\xi) \to B$.

the unit sphere bundle

$\xi$ has a nonvanishing section $\Leftrightarrow$ $S(\xi)$ has a section.

Fiber of $S(\xi)$ is $(n-1)$-connected

first obstruction to a section of $S(\xi)$ is

$$o_n \in H^n(B, \underset{\mathbb{Z}}{\underset{\cong}{\pi_{n-1}(S^{n-1},)}})$$

**Thm:** If $\xi$ is oriented, then $\pi_1 B \curvearrowright \mathbb{Z}$ trivially.

$$\dot{o}_n = e(\xi) \in H^n(B, \mathbb{Z})$$

Euler class $\uparrow$ trivial coefficient.

(1) $\pi_1 B \curvearrowright \pi_{n-1}(S^{n-1})$ trivially since

**Rmk:** $f: S^{n-1} \to S^{n-1}$ orientation preserving homeo $\Rightarrow$ $\deg = 1$.

If $\xi$ is not orientable, we have

$$ob \in H^n(B, \tilde{\mathbb{Z}}).$$

or

$$\underset{\substack{\parallel \\ \omega_n(\xi)}}{ob \ (\mathrm{mod}\ 2)} \in H^n(B, \mathbb{Z}/2\mathbb{Z}).$$

(2) When $n=2$.

$$\pi_i(S^1) = 0 \qquad \forall i \neq n-1.$$

so $H^{i+1}(B, \pi_i(S^1)) = 0 \qquad \forall i \neq n-1$

$\nexists$ nontrivial obstruction beyond the first ob.

**Prop:** An oriented rank 2 real vector bundle $\xi$ has a nonvanishing section

$\Longleftrightarrow$ $e(\xi) = 0$

**Cor:**

$$\begin{array}{ccc} \mathrm{Vect}^1_{\mathbb{C}}(B) & \longrightarrow & H^2(B;\mathbb{Z}). \text{ is injective.} \\ \parallel & \xi \longmapsto & c_1(\xi) = e(\xi). \\ \{\text{complex line bundles over } B\} & & \end{array}$$

In fact, $\hookrightarrow$ is also $\twoheadrightarrow$.

$$\begin{tikzcd}
\mathrm{Vect}^1_{\mathbb{C}}(B) \arrow[r, "\cong"] \arrow[d, equal] & H^2(B,\mathbb{Z}) \arrow[d, equal] \\
{[B, BU_1]} \arrow[r] & {[B, K(\mathbb{Z},2)]}
\end{tikzcd}$$

$$BU_1 \simeq \mathbb{C}P^\infty \simeq K(\mathbb{Z},2).$$

(your HW).

(3) These result fails in general for $n>2$ since $S^{n-1}$ might have nontrivial $\pi_i$ where $i>n-1$.

e.g. $n=3$. $\pi_3(S^2)\cong\mathbb{Z}$

so even if the first ob. $=0$ there might be nontrivial obstruction afterwards in

$$H^{i+1}(B, \pi_i(S^{n-1})) \quad \text{when } i>n-1.$$

pf sketch: First we claim $O_n(\xi) = \lambda \cdot e(\xi)$ for some $\lambda \in \mathbb{Z}$.

reason:

$$\begin{tikzcd}
\pi_0^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\
\xi_0 \arrow[r, "\pi_0"] & B
\end{tikzcd}$$

$\pi_0 \xi$ has a nonzero section. (tautological)

so $\pi_0^* O_n(\xi) = 0$

Gysin

$$H^0(B) \xrightarrow{\cup e} H^n(B) \xrightarrow{\pi_0^*} H^n(\xi_0)$$

$$\lambda \longmapsto O_n(\xi) = \lambda \cup e(\xi).$$

Next, we compute $\lambda = 1$ on the universal (oriented) bundle.

$\tilde{G}_n := \{$oriented n-planes in $\mathbb{R}^\infty\}$.

$\tilde{G}_n \xrightarrow{2:1} G_n$ cover.

$\tilde{\gamma}_n$ universal oriented bundle.

$$\begin{tikzcd}
\tilde{\gamma}_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \gamma_n \arrow[d] \\
\tilde{G}_n \arrow[r] & G_n
\end{tikzcd}$$

$$\begin{tikzcd}
 & \tilde{G}_n \arrow[d] \\
B \arrow[ur, dotted] \arrow[r] & G_n
\end{tikzcd}$$

$[B, \tilde{G}_n] \cong \{$oriented n-plane bun over B$\}/\cong$

We see $\lambda \in \mathbb{Z}$ the same for all bundle $\xi$.

when $n =$ even. take $\xi = \tau_{S^n}$.

explicit computation (same as last time):

$\Rightarrow \langle O_n(\tau_{S^n}), [S^n] \rangle = +2.$

$\Rightarrow O_n(\tau_{S^n}) = 1 \cdot e(\tau_{S^n})$

$\Rightarrow \lambda = 1.$

when $n =$ odd. ∃ an isomorphism of bundle $v \mapsto -v$ is orientation reversing.

~~note~~ $\Rightarrow e(\xi) = -e(\bar{\xi}) = -e(\xi).$

$\Rightarrow 2e(\xi) = 0.$

To show $O(\xi) = e(\xi)$

it suffices to prove the statement (mod 2).

**Thm**: $H^\bullet(\tilde{G}_n ; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[w_2, \dots, w_n]$

$$w_i = w_i(\tilde{\gamma}_n)$$

**Pf sketch**: $\underset{\mathbb{R}^1}{\underset{\cap}{S^0}} \to \tilde{G}_n \xrightarrow{p} G_n$.

Gysin sequence mod 2: $\underset{\underset{u(e \bmod 2)}{\|}}{w_1(p) \neq 0}$. otherwise $p$ is trivial cover. $\tilde{G}_n$ disconnected. ↯.

$$H^{j-1}(G_n) \xrightarrow{u(e \bmod 2)} H^j(G_n) \xrightarrow{p^*} H^j(\tilde{G}_n) \to H^j(G_n) \to \cdots$$

$$w_1(\gamma_n) \longmapsto w_1(\tilde{\gamma}_n) = 0$$

since $w_1(\gamma_n) \in \underset{\underset{\ker p^*}{\|}}{\text{image}(\cup w_1(\gamma_n))}$. $\square$.

**Thm** $\Rightarrow$ (mod 2):

$$o_n(\tilde{\gamma}_n) = w_n(\tilde{\gamma}_n) = \lambda e(\tilde{\gamma}_n) = e(\tilde{\gamma}_n)$$

$\Rightarrow \lambda = 1 \bmod 2$.

$\square$

# principal G-bundles.

$G =$ a topological group.

A **principal G-bundle** is a locally trivial fiber bundle

$$\pi : P \longrightarrow B$$

together with a continuous right action $P \times G \longrightarrow P$ s.t. G acts freely and transitively on each fiber.

Hence, $$G \xrightarrow{\cong} \pi^{-1}(b)$$
$$g \longmapsto x_0 \cdot g \qquad x_0 \in \pi^{-1}(b).$$

Sometimes we write $G \to P \to B$.

Local triviality $\Rightarrow B \cong P/G$

Rmk: Every right action $X \circlearrowleft G$

gives a left action $G \circlearrowright X$

by $g \cdot x := x g^{-1}$

and vice versa.

**EXO**: Cover. (Ex3).

**Ex1**: $\xi$ a real vector bundle of dim $= n$.

$$\xi_b \to E \xrightarrow{\pi} B.$$

Consider the $n$-frame bundle: $F(\xi)$.

$$F(\xi_b) \to F(E) \longrightarrow B.$$

where $F(\xi_b) := \{(v_1 \cdots v_n) \in \xi_b \mid \text{independent}\}$

$$\cong \mathrm{Iso}(\mathbb{R}^n, \xi_b)$$

$$\circlearrowleft$$

$$\mathrm{Iso}(\mathbb{R}^n, \mathbb{R}^n) = GL_n(\mathbb{R}).$$ freely and transitively

$F(\xi)$ is a principal $GL_n(\mathbb{R})$-bundle.

**Observe**: There is a bijection

$$\left\{\begin{matrix}\text{rank-}n\text{ vector}\\ \text{bundles over } B\end{matrix}\right\}\Big/\cong \;\longleftrightarrow\; \left\{\begin{matrix}\text{principal}\\ GL_n(\mathbb{R})\text{-bundles}\\ \text{over } B\end{matrix}\right\}\Big/\cong$$

$$\xi \longmapsto F(\xi).$$

Inverse: Given a principal $GL_n(\mathbb{R})$-bundle

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d, phantom, "\circlearrowright"] & B \\ GL_n(\mathbb{R}). & \end{tikzcd}$$

form:

$$\begin{tikzcd} E := P \times_{GL_n(\mathbb{R})} \mathbb{R}^n \arrow[d, "p_1"] \\ P/GL_n = B \end{tikzcd}$$

$E \xrightarrow{p_1} B$ is a vector bundle of rank $n$.

Similarly, suppose $\xi$ has an Euclidean metric, let

$$OF(\xi_b) := V_n(\xi_b) = \{(v_1 \cdots v_n) \mid \text{orthonormal}\}$$

$$\cong \text{Isometry}(\mathbb{R}^n, \xi_b)$$

$$\circlearrowright$$

$$O_n$$

Stiefel manifold.

$OF(\xi)$ is a principal $O_n$-bundle.

$$\left\{ \begin{array}{l} \text{rank-n vector} \\ \text{bundles with} \\ \text{a E. m.} \end{array} \right\} \Big/ \cong \quad \overset{\cong}{\longleftrightarrow} \quad \left\{ \begin{array}{l} \text{principal} \\ O_n\text{-bun} \end{array} \right\} \Big/ \cong .$$

Similar story if
$$\mathbb{R} \rightsquigarrow \mathbb{C}$$
$$O_n \rightsquigarrow U_n$$
Euclidean ⇝ Hermitian.

**Ex 2:** $B = G_n(\mathbb{R}^{n+k})$

$P = V_n(\mathbb{R}^{n+k})$

$$\begin{tikzcd} V_n(\mathbb{R}^{n+k}) \arrow[r] & G_n(\mathbb{R}^{n+k}) \end{tikzcd} \quad \text{principal } O_n\text{-bundle} = OF(\gamma^n_{n+k})$$

$$(v_1, \cdots, v_n) \longmapsto \operatorname{span}\{v_1, \dots v_n\}.$$

$$\begin{tikzcd} O_n \arrow[r] \arrow[d, equal] & V_n \arrow[r] \arrow[d, equal] & G_n \arrow[d, phantom, "\cong" rotate=90] \\ O_n \arrow[r] & \dfrac{O_{n+k}}{O_k} \arrow[r] & \dfrac{O_{n+k}}{O_n \times O_k} \end{tikzcd}$$

this bundle is exactly $OF(\gamma_n^k)$.

**Ex3:** $G$ discrete top.

Then a principal $G$-bundle

$= $ a normal cover $\tilde{B} \xrightarrow{\pi} B$

s.t. $\pi_1 \tilde{B} \trianglelefteq \pi_1 B$

and $\dfrac{\pi_1 B}{\pi_1 \tilde{B}} \cong G$. (Galois cover).

**Ex4:**

$$\begin{tikzcd} \mathbb{C} & \mathbb{C} \\ S^1 \arrow[u, phantom, "\subseteq" sloped] \arrow[r] & S^1 \arrow[u, phantom, "\subseteq" sloped] \\ z \arrow[r, mapsto] & z^d \end{tikzcd}$$

is a principal $\mu(d)$-bundle

$$\mu(d) = \{\lambda \mid \lambda^d = 1\} \cong \mathbb{Z}/d\mathbb{Z}.$$

**Ex5:**

$$\begin{tikzcd} S^0 & & \\ O(1) \arrow[u, phantom, "\cong" sloped] \arrow[r] & S(\mathbb{R}^{n+1}) \arrow[r] & \mathbb{RP}^n \\ S^1 \cong U(1) \arrow[r] & S(\mathbb{C}^{n+1}) \arrow[r] & \mathbb{CP}^n \\ Sp(1) \arrow[r] \arrow[d, phantom, "\cong" sloped] & S(\mathbb{H}^{n+1}) \arrow[r] & \mathbb{HP}^n \\ SU(2) \cong S^3. & & \end{tikzcd}$$

unit quaternions

non-Example:

$\mathbb{Q} \to \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is not a principal $\mathbb{Q}$-bundle.

$\mathbb{R}/\mathbb{Q}$ has trivial topology.

locally trivial $\Rightarrow$ globally trivial

$\Rightarrow \mathbb{R} \cong \mathbb{Q} \times \mathbb{R}/\mathbb{Q}$ ↯.

Ex 6: $G$ a Lie group

$H$ closed subgroup (not necessarily normal)

$$\begin{tikzcd} H \arrow[r] & G \arrow[d] \\ & G/H \end{tikzcd}$$

is a principal $G$-bundle.

$G/H \to$ "homogeneous space".

e.g.

$$\begin{tikzcd} O(n) \arrow[r] & O(n+1) \arrow[d] \\ & \frac{O(n+1)}{O(n)} \cong S^n \subseteq \mathbb{R}^{n+1} \end{tikzcd}$$

principal $O(n)$-bundle.

$U(n) \longrightarrow U(n+1) \longrightarrow S^{2n+1} \subseteq \mathbb{C}^{n+1}$

In general, $G$ Lie group $\curvearrowright M$ smooth manifold

action is smooth and free.

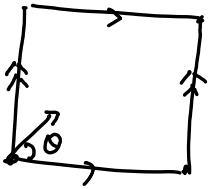
$M \to M/G$ might not be a p.G.b.

counterexample: $M = T^2$.

$G = \mathbb{R}$.

$\mathbb{R} \curvearrowright T^2$ by

$T^2 = \mathbb{R}^2/\mathbb{Z}^2$



$\theta = \alpha \cdot \pi \quad \alpha \notin \mathbb{Q}$.

However, $M \to M/G$ is a pGb

if $G$ is compact

.

**Prop**: A principal $G$-bundle $\pi: P \to B$ is trivial iff it admits a section.

**Rmk**: This is false for fiber bundles in general.

counter examples. • vector bundles always have zero sections.

• $S^n$ is parallelizable iff $n = 0, 1, 3, 7$. (Bott, Kervaire, Milnor).

$S^n$ has a nonvanishing vector field iff $n$ is odd.

$$\mathbb{R}^n \supseteq S^{n-1} \to UT(S^n) \rightleftarrows S^n$$

$n$ odd, $n \neq 1, 3, 7$.

$\exists$ section. but not trivial.

**pf** sketch: Suppose
$$\begin{tikzcd} P \arrow[r, "\pi"'] & B \arrow[l, bend right, "s"'] \end{tikzcd}.$$

Define
$$\begin{tikzcd} B \times G \arrow[r, "f"] & P \end{tikzcd}$$
$$(b, g) \longmapsto s(b) \cdot g$$

Clearly. $f$ is cont' and a bijection. $(B = P/G)$.

To show $f^{-1}$ is continuous. use local trivialization.

$$U \times G \longrightarrow \pi^{-1}(U)$$

**Def:** We say $BG$ is a classifying space for $G$, a top. group

if $\exists$ a principal $G$-bundle $EG \longrightarrow BG$

s.t. $EG$ is weakly contractible

i.e. $\pi_i(EG) = 0 \quad \forall i$

**Thm:** For any CW complex $B$, the map

$$\begin{aligned} [B, BG] &\longrightarrow \mathrm{Bun}_G(B) = \left\{ \begin{matrix} \text{p. G. b.} \\ \text{over } B \end{matrix} \right\} \Big/ \cong \\ \phi &\longmapsto \phi^*(EG) \end{aligned}$$

$$\begin{tikzcd} G \arrow[d] & G \arrow[d] \\ \phi^* EG \arrow[r] \arrow[d] & EG \arrow[d] \\ B \arrow[r, "\phi"] & BG \end{tikzcd}$$

is a bijection.

Say: $EG \to BG$ is a universal $G$-bundle.

$BG$ classifies principal $G$-bundles

<u>Pf.</u> of Thm:

(onto): Given
$$\begin{tikzcd} G \arrow[d] \\ P \arrow[d] \\ B \end{tikzcd}$$

form:
$$\begin{tikzcd} EG \arrow[r] & P \times_G EG \arrow[d, shift right] \\ & P/G = B \arrow[u, dotted, shift right, "\exists s"'] \end{tikzcd}$$

section exists because $\underset{\overset{\|}{0}}{ob_{i+1}} \in H^{i+1}(B, \pi_i(EG))$

**note**: B is CW.

$$s : P/G \longrightarrow P \times_G EG$$

$$p \longmapsto (p, \underset{\in EG}{\tilde{\phi}(p)})$$

$$p \cdot g \longmapsto (p \cdot g, \tilde{\phi}(p \cdot g))$$
$$\tilde{\phi}(p \cdot g) = \tilde{\phi}(p) \circ g$$

so $\tilde{\phi} : P \to EG$ is G-equivariant.

Then
$$\begin{tikzcd} P \arrow[r, "\tilde{\phi}"] \arrow[d] & EG \arrow[d] \\ B = P/G \arrow[r, "\phi"] & EG/G = BG. \end{tikzcd}$$

check this is a pullback.

(1-1): Suppose $f_0, f_1 : B \longrightarrow BG$

s.t. $f_0^* EG \xrightarrow[\cong]{\psi} f_1^* EG = P$

each $f_0, f_1$ gives sections of

$$\begin{tikzcd}
EG \arrow[r] & P \overset{\times}{\underset{G}{}} EG \arrow[d] \\
& P/G = B \arrow[u, dashed, bend right, "s_0"'] \arrow[u, dashed, bend right=60, "s_1"']
\end{tikzcd}$$

obstruction to a homotopy $s_0 \sim s_1$ lives in

$$H^{i+1}(B \times I, B \times \partial I, \pi_i(EG)) = 0.$$

so $s_0 \simeq s_1$

$$\begin{tikzcd}
P \arrow[r, "\tilde{\phi}_0 \simeq \tilde{\phi}_1"] \arrow[d] & EG \arrow[d] \\
B \arrow[r, "\phi_0 \simeq \phi_1"] & BG
\end{tikzcd}$$

$\square$

**Rmk**: ① BG can be taken to be a CW complex.

( if not, take $(BG)'$ a CW complex

and $BG' \xrightarrow{f} BG$ a weak equivalence.

then $f^*EG \underset{\text{weak}}{\simeq} EG$ weakly contractible.)

② $G \to EG \to BG \Rightarrow \pi_i(BG) \cong \pi_{i-1}(G)$.

③ BG is unique up to homotopy equivalence.

(HW: use obstru. th. or Yoneda's lemma).

$\Rightarrow$ BG is unique up to weak h.e.

for CW complexes, h.e. = weak h.e. )

EG is unique up to G-homotopy equivalence.

Sometimes we say BG is "the" classifying space for G.

Say: Unique up to w.h.e.

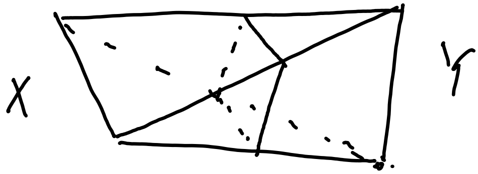
or up to h.e. for CW complexes.

④ (Milnor).

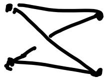
BG exists for any topological group G.

# Milnor's construction of BG (sketched).

Join: $X * Y := X \times [0,1] \times Y \Big/ \begin{matrix} (x,0,y_1) \sim (x,0,y_2) \\ (x_1,1,y) \sim (x_2,1,y) \end{matrix}$



Ex: $S^0 * S^0 = S^1$



$S^n * S^m = S^{n+m+1}$

$(S^0)^{*n} = S^{n-1}$.

Fact: For $X$ any space, $X^{*n}$ is $(n-2)$-connected.

Milnor:

$$G \subseteq G^{*2} \subseteq \cdots \subseteq G^{*n} \subseteq \cdots$$

$EG := \bigcup_n G^{*n}$ weakly contractible

$G \curvearrowright EG$ freely.

$BG := EG/G$.

(Not if $G$ is CW complex, so is $BG$.)

STOP

Say: If $G$ is a CW complex, so is Milnor's model for $BG$.

Prop: A map $H \xrightarrow{f} G$ of top. groups induces a map $BH \xrightarrow{Bf} BG$ of spaces that is unique up to homotopy.

Hence, $G \longmapsto BG$ is a functor from the cat. of top. groups to the homotopy cat. of CW complexes.

Pf: Given $H \xrightarrow{f} G$, form

$$\begin{tikzcd} G \arrow[r] & EH \times_H G \arrow[d] \\ & EH/H = BH \end{tikzcd}$$

a principal $G$-bundle.

⇝ classifying map $BH \longrightarrow BG$. unique up to homotopy.

**Cor**: If $H \xrightarrow{f} G$ is a homotopy equivalence then $BH \xrightarrow{Bf} BG$ is also a h.e.

**Pf**:

$$\begin{tikzcd}
\pi_i(BH) \arrow[r, "Bf_*"] \arrow[d, phantom, "\cong" rotate=90] & \pi_i(BG) \arrow[d, phantom, "\cong" rotate=90] \\
\pi_{i-1}(H) \arrow[r, "f_*"] & \pi_{i-1}(G)
\end{tikzcd}$$

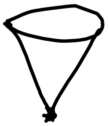
$\square$.

**Ex 1**: $O(n) \hookrightarrow Gl_n(\mathbb{R})$ is a h.e.

**pf 1**: Gram-Schmidt : $Gl_n(\mathbb{R}) \simeq O(n)$

**pf 2**: $Gl_n(\mathbb{R}) / O(n) \cong \{$ Euclidean metric $\langle , \rangle$ on $\mathbb{R}^n \}$.

$\cong \{ S \in M_n(\mathbb{R}) \mid$ positive definite symmetric $\}$.



convex cone : $tS + (1-t)S'$ "

$\simeq *$

Hence, $BO(n) = BGl_n(\mathbb{R})$.

**Rmk**: When I write $BG = BG'$, I mean $BG = BG'$ in the homotopy cat. of CW complexes.

Similarly. $BU(n) = BGl_n(\mathbb{C})$.

<u>Ex 2</u>: $\mathbb{R}^n \cong 1$ trivial group.

$$\Rightarrow B\mathbb{R}^n = B1 = *$$

Every principal $\mathbb{R}^n$- bundle is trivial.

## Examples of explicit models for $BG$.

(0) $G$ discrete. $BG = K(G,1)$.

Say: "$G$- bundle theory = covering space theory reduced to $\underbrace{\text{group}}_{\text{discrete}}$ theory".

① $E\mathbb{Z} = \mathbb{R}$

$B\mathbb{Z} = \mathbb{R}/\mathbb{Z} = S^1$

② $EU(1) = S^\infty \subseteq \mathbb{C}^\infty \setminus 0$ . $(S^\infty \simeq *)$.

$BU(1) = S^\infty/U(1) = \mathbb{C}P^\infty$.

**note**: $O_1 = \{\pm 1\} \subseteq U(1) \curvearrowright S^\infty$

hence $EO_1 = S^\infty$

$BO_1 = S^\infty/C_2 = \mathbb{R}P^\infty$.

In general, $BC_n = S^\infty/C_n$ $\infty$-dim lens space.

where $C_n \subseteq U(1) \curvearrowright S^\infty$

③ Recall

$$O(n) \longrightarrow V_n(\mathbb{R}^{n+k}) \longrightarrow G_n(\mathbb{R}^{n+k}).$$

↳ $\pi_i = 0 \quad \forall\, i<k.$

Let $k \to \infty$, $V_n := V_n(\mathbb{R}^\infty)$ is weakly contractible.

Hence, $V_n = EO(n)$

$G_n = BO(n)$

observe, if we replace $V_n$ by $V_n'$

$$V_n' := \{ (v_1, \cdots, v_n) \in \mathbb{R}^\infty \,/\, \text{independent} \}.$$

$\circlearrowleft$

$GL_n(\mathbb{R})$.

so $EGL_n(\mathbb{R}) = V_n'$

$BGL_n(\mathbb{R}) = V_n'/GL_n(\mathbb{R}) = G_n = BO_n.$

<u>Fact</u>: $GL_n(\mathbb{R}) \rightsquigarrow O(n)$.

Gram-Schmidt process

$B$ = CW complex.

$BO(n) = G_n$

$$\begin{tikzcd}
 & \left\{\begin{array}{l}\text{n-dim vector bundles} \\ \text{over } B \text{ with an} \\ \text{Euclidean metric}\end{array}\right\} \arrow[dl, leftrightarrow] \arrow[dd] \arrow[dr, leftrightarrow] & \\
\left\{\begin{array}{l}\text{principal } O(n)\text{-bundles} \\ \text{over } B\end{array}\right\}\Big/\cong \arrow[rr, phantom, "="] \arrow[dd] & & {[B, BO(n)]} \arrow[dd] \\
 & \left\{\begin{array}{l}\text{n-dim vector bundles} \\ \text{over } B\end{array}\right\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\left\{\begin{array}{l}\text{principal } Gl_n(\mathbb{R})\text{-bundles} \\ \text{over } B\end{array}\right\}\Big/\cong \arrow[rr, phantom, "="] & & {[B, BGl_n(\mathbb{R})]}
\end{tikzcd}$$

Similar story for $\mathbb{C}$:

$$BU(n) = G_n(\mathbb{C}^\infty) = BGl_n(\mathbb{C}).$$

## Change of structure group.

Many questions about a vector bundle $\xi$

(e.g. orientable? Euclidean metric?

$\exists$? nonvanishing section

sum of two subbundles? ... )

can be formulated in a uniform way about change of structure group.

**prop:** Suppose $G$ is a Lie group.

$H \leq G$ a closed subgroup

(so $H \to G \to G/H$ is a fibration).

$P \to B$ a principal $G$-bundle over a CW complex $B$.

Then TFAE.

(a) $P$ is induced from an $H$-bundle

i.e. $\exists$ a $H \to Q \to B$ s.t. $P = Q \times_H G$.

(say: extension of group action).

(b) $P \times_G (G/H) \to B$ has a section.

(c) The classify map of $P$ lifts to $BH$ up to homotopy:

$$\begin{tikzcd} & BH \arrow[d, "Bi"] \\ P \arrow[r] \arrow[ur, dashed] & BG \end{tikzcd}$$

**note:** $\exists$ a fibration $G/H \to BH \xrightarrow{Bi} BG$.

$$\begin{tikzcd} G/H \arrow[r] & BH \arrow[r, "Bi"] \arrow[d, phantom, "\parallel"] & BG \arrow[d, phantom, "\wr"] \\ G/H \arrow[r] & EG/H \arrow[r] & EG/G \end{tikzcd}$$

upshot: The viewpoint of $BG$ allows us to rethink vect. bundle theory.

Example 1: $H = O(n)$

$G = Gl_n(\mathbb{R})$.

$\mathbb{R}^n \to E \to B$ a vector bundle

$Gl_n(\mathbb{R}) \to P \to B$ the associated frame bundle.

(a) says:

$$\begin{tikzcd}
Gl_n \arrow[r] & P \arrow[r] & B \\
O(n) \arrow[r] \arrow[u] & Q \arrow[r] \arrow[u] & B \arrow[u, equal]
\end{tikzcd}$$

There is a notion of "orthonormal" on $E$.

(b) says:

$$\begin{tikzcd}
P \times_{Gl_n} Gl_n/O(n) \arrow[r] & B \arrow[l, dashed, bend right, "s"']
\end{tikzcd}$$

has a sections

note: $Gl_n/O(n) = \{$ Euclidean metrics on $\mathbb{R}^n \}$.

so a section $s$ gives an Euclidean metric on $E$.

(c) says:

$$\begin{tikzcd}
 & BO(n) \arrow[d] \\
B \arrow[r] \arrow[ur, dashed, "\exists"] & BGl_n(\mathbb{R})
\end{tikzcd}$$

(a) (b) (c) holds for any $P \to B$. $B = CW$

Same story for $U(n) \hookrightarrow Gl_n(\mathbb{C})$.

**Ex 2:** $G = O(n)$ $\qquad$ $H = SO(n)$.

$$O(n)/SO(n) \xrightarrow{\det} \{\pm 1\}.$$

$G \to P \to B$ is the frame bundle of a vector bundle $\xi$.

prop $\Rightarrow$ $\xi$ is orientable iff

$$\begin{tikzcd} & BSO(n) \arrow[d] \\ B \arrow[r, "f_\xi"] \arrow[ur, dashed] & BO(n) \end{tikzcd}$$

note:

$$\begin{tikzcd} \frac{O(n)}{SO(n)} \arrow[r] \arrow[d, equal] & BSO(n) \arrow[r, "2:1"] \arrow[d, equal] & BO(n) \arrow[d, equal] \\ \{\pm 1\} & \tilde{G}_n & G_n \end{tikzcd}$$

the only obstruction to a lift is in

$$\underset{\substack{\uparrow \\ [\text{why? } O_1 \neq 0]}}{O_1 = w_1} \in H^1(BO(n);\ \underset{\substack{\| \\ \mathbb{Z}/2\mathbb{Z}}}{\pi_0(\{\pm 1\})}) = Hom(\underset{\substack{\| \\ \pi_0 O(n) \\ \| \\ \mathbb{Z}/2\mathbb{Z}}}{\pi_1 BO(n)}, \{\pm 1\})$$

lift exists $\Leftrightarrow$ $f_\xi^* w_1 = w_1(\xi) = 0$.

Conclude: $\xi$ orientable iff $w_1(\xi) = 0$.

**Ex:**

$$G = U(n)$$

$$H = T^n = \left\{ \begin{bmatrix} * & & \\ & \ddots & \\ & & * \end{bmatrix} \right\} = U(1)^{\times n}.$$

$$\begin{tikzcd} \frac{U(n)}{T^n} \arrow[r] & BT^n \arrow[r, "p"] \arrow[d, phantom, "\circlearrowleft"] & BU(n) \\ & S_n. & \end{tikzcd}$$

$$\frac{U(n)}{T^n} = \{(L_1, \dots, L_n) \mid L_i \text{ orthogonal in } \mathbb{C}^n\}.$$

"flag manifold".

**Fact:** Leray-Hirsch applies:

$$\begin{tikzcd} H^\bullet(BT^n) \arrow[d, phantom, "\circlearrowleft"] \arrow[r, phantom, "\cong"] & H^\bullet(BU(n)) \otimes H^\bullet\left(\frac{U_n}{T^n}\right) \\ S_n. & \end{tikzcd}$$

as $H^\bullet(BU(n))$-modules.

$$\begin{tikzcd} p^*: H^\bullet(BU(n)) \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & H^\bullet(BT^n) \arrow[d, phantom, "\shortparallel"] \\ \mathbb{Z}[x_1, \dots, x_n]^{S_n} \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & \mathbb{Z}[x_1, \dots, x_n] \\ \mathbb{Z}[c_1, \dots, c_n] & \end{tikzcd}$$

$c_i = \pm S_i(x_1 \cdots x_n).$

**Splitting principle**: Given $\mathbb{C}^n$-bundle $\xi$ over $B$.

$$\begin{tikzcd}
U_n/T(n) \arrow[d] & U_n/T(n) \arrow[d] \\
B' := f^*BT(n) \arrow[r] \arrow[d, "f^*p"'] & BT(n) \arrow[d, "p"] \\
B \arrow[r, "f"] & BU(n)
\end{tikzcd}$$

consider $f^*B(T(n)) =: B' \longrightarrow B$

① The pullback of $\xi$ to $B'$ is a sum of line bundles.

② Leray - Hirsch applies to $\frac{U_n}{T(n)} \to B' \to B$.

so $H^*(B') \cong H^*(B) \otimes H^*(U_n/T(n))$

in particular $H^*(B) \hookrightarrow H^*(B')$.

$B' = f^*BT(n) = \{(L_1, \dots L_n; b) \mid b \in B,$

$L_i$'s are orthonormal $\xi$

## Manifold bundles.

Vector bundle theory ( fiber = vector space )

$$\wr\!\downarrow$$

manifold bundle theory ( fiber = $M^n$ manifolds).

As before, we have:

$$G = \mathrm{Diffeo}(M^n) := \{ \text{diffeo } M^n \xrightarrow{f} M^n \}$$

$$\left\{ \begin{array}{l} \text{smooth bundles} \\ F \to E \to B \\ F \cong M^n. \end{array} \right\} \Big/ \cong \quad \longleftrightarrow \quad \left\{ \begin{array}{l} \text{principal} \\ \mathrm{Diffeo}(M^n)\text{-bundles} \\ \text{over } B \end{array} \right\} \Big/ \cong$$

$$\updownarrow$$

$$[B, \; B\mathrm{Diffeo}(M^n)]$$

$$(F \to E \to B) \longmapsto (\mathrm{Diffeo}(F, M^n) \to P \to B).$$

$$\left( M^n \to P \times_G M^n \to \frac{P}{G} = B \right) \mathrel{\reflectbox{$\longmapsto$}} (G \to P \to B)$$

$B\mathrm{Diffeo}(M)$ = is called "the moduli / classifying space of $M$ - bundles".

(I) $M^n = S^1$. "circle bundles".

$$U(1) = \{\text{rotations}\} \subseteq \mathrm{Diffeo}^+(S^1)$$

$\overset{\shortparallel}{SO(2)}$.

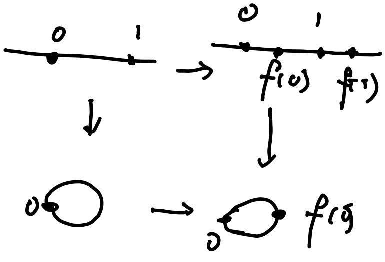
"+" means orientation-preserving.

<u>prop</u>: $\mathrm{Diffeo}^+(S^1) \simeq U(1)$

(exercise).

<u>idea</u>: Given $f: S^1 \to S^1$ lift to $\mathbb{R}$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d] & \mathbb{R} \arrow[d] \\ S^1 \arrow[r, "f"] & S^1 \end{tikzcd}$$



Hence: $B\mathrm{Diffeo}^+(S^1) = BU(1) = \mathbb{C}P^\infty = K(\mathbb{Z},2)$.

$$\left\{\begin{matrix}\text{orientable} \\ S^1\text{-bundles} \\ \text{over } B\end{matrix}\right\} \Big/ \cong \quad \overset{\cong}{\longleftrightarrow} [B, BU(1)] \overset{\cong}{\longleftrightarrow} H^2(B,\mathbb{Z}).$$

<u>Rmk</u>: (1) Every $S^1$-bundle is (iso. to) the unit of an rank-2 real vector bundle.

(2) Euler class is the complete invariant.

(II) $M^n = S^n$.

$SO(n+1) \hookrightarrow \mathrm{Diffeo}^+(S^n)$ is a weak homotopy equivalence when

$n=2$ (Smale; 1959)
$n=3$ (Hatcher, 1983).

$\Rightarrow$ $B\mathrm{Diffeo}^+(S^n) = BSO(n+1)$.

"$S^n$- bundle theory can be reduced to vector bundle theory".

It is not a w.h.e. when

$n \geqslant 5$ (Kervaire-Milnor 1963)
$n=4$ (Watanabe, 2018).

"Smale's conjecture".

(III) $M^n = T^2 = S^1 \times S^1$.

$$\left\{ \begin{matrix} T^2\text{-bundles} \\ \text{over } B \end{matrix} \right\} \longleftrightarrow [B, B\mathrm{Diff}(T^2)].$$

$$\left\{ \begin{matrix} T^2\text{-bundles over } B \\ \text{with a section} \end{matrix} \right\} \longleftrightarrow [B, B\mathrm{Diff}(T^2, 0)]$$

<u>prop.</u> $\underset{\overset{\circlearrowright}{\mathbb{R}^2/\mathbb{Z}^2}}{SL(2,\mathbb{Z})} \hookrightarrow \mathrm{Diff}^+(T^2, 0)$ is a h.e.

<u>Cor</u>: $B\mathrm{Diff}^+(T^2,0) \simeq BSL(2,\mathbb{Z})$.

<u>note</u>: $B\mathrm{Diff}^+(T^2,0) \simeq BSL(2,\mathbb{Z}) \longrightarrow BSL(2,\mathbb{R}) \simeq \underset{\overset{\wr\|}{BU(1)}}{BSO(2)}$

$\Downarrow$

$\exists$ a map

$$\left\{ \begin{matrix} \text{orientable} \\ T^2\text{-bund.} \\ \text{w/ a section} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{circle} \\ \text{bundle} \end{matrix} \right\}$$

<u>Q</u>: What is this map? <u>exercise</u>.

(III) $M^n = \Sigma_g$. $(g \geq 2)$. "Surface bundles".

"simplest nonlinear theory"?

$$\left\{ \begin{array}{c} \text{smooth} \\ \Sigma_g\text{-bundles} \\ \text{over } B \end{array} \right\} \Big/ \cong \quad \longleftrightarrow \quad [B, B\mathrm{Diff}(\Sigma_g)]$$

<u>Earle-Eells (1967)</u>*: $\mathrm{Diff}_0(\Sigma_g)$ is contractible

<u>Def</u>. The <u>mapping class group</u> of $\Sigma_g$ is

$$\mathrm{Mod}(\Sigma_g) := \frac{\mathrm{Diff}^+(\Sigma_g)}{\mathrm{Diff}_0(\Sigma_g)} \quad \text{discrete group.}$$

$$EE \Rightarrow \quad B\mathrm{Diff}^+(\Sigma_g) \simeq B\mathrm{Mod}(\Sigma_g).$$

$$\left\{ \begin{array}{c} \text{smooth orientable} \\ \Sigma_g\text{-bundles} \\ \text{over } B \end{array} \right\} \Big/ \cong \quad \leftrightarrow \quad [B, B\mathrm{Mod}(\Sigma_g)].$$

$$H^*(B\mathrm{Mod}(\Sigma_g)) = \left\{ \begin{array}{c} \text{characteristic classes} \\ \text{of surface bundles} \end{array} \right\}.$$

Fact: $H^\bullet(B\mathrm{Mod}(\Sigma_g);\mathbb{Q}) \cong H^\bullet(\mathcal{M}_g;\mathbb{Q})$

$\mathcal{M}_g$ = moduli space of Rieman surfaces of genus $g$.

Thm. (Harer, 1985).

$(k < 2\lfloor \frac{g}{3} \rfloor)$.

$H^k(B\mathrm{Mod}(\Sigma_g);\mathbb{Z})$ stabilizes as $g \to \infty$.

"Homological stability"

Thm (Madsen-Weiss 2006)

$H^\bullet(B\mathrm{Mod}(\Sigma_g);\mathbb{Q}) \longrightarrow \mathbb{Q}[\kappa_1, \kappa_2, \cdots]$ as $g \to \infty$.

$\kappa_i \in H^{2i}$. "Morita-Mumford-Miller classes" or tautological classes.

Thm. (Harer-Zagier, 1986)

$$\chi(\mathcal{M}_g) = \chi(B\mathrm{Mod}(\Sigma_g)) = \frac{\zeta(1-2g)}{2-2g}$$

$$\sim (-1)^g \frac{(2g-1)!}{(2-2g)\, 2^{2g-1}\pi^{2g}} \quad \text{as } g \to \infty.$$

<u>Rmk</u>: As $g \to \infty$.

$\chi(BMod(\Sigma_g))$ grows superexponentially in $g$

However, # of $\underbrace{\text{stable classes}}_{\{k_i\}}$ grows polynomially in $g$.

"Dark matter problem": (open).

Find some (or even one!) ~~[illegible]~~ element in $H^k(BMod(\Sigma_g))$ outside of the stable range.

**Next**: Pontrjagin classes

applications: Thom's cobordism theorem
Hirzebruch's signature thm
Milnor's exotic spheres.

Reference: Milnor - Stasheff §15.

**Linear algebra**: $V$ a real vector space of $\dim_{\mathbb{R}} n$

$V \otimes_{\mathbb{R}} \mathbb{C}$ a complex v.s. of $\dim_{\mathbb{C}} n$.

$$\begin{gathered} V \otimes_{\mathbb{R}} \mathbb{C} \\ \parallel \\ \{ x+iy \mid x, y \in V \}. \end{gathered}$$

"complexification"

**note**:

$$\begin{aligned} V \otimes_{\mathbb{R}} \mathbb{C} &\longrightarrow \overline{V \otimes_{\mathbb{R}} \mathbb{C}} \\ x+iy &\longmapsto x-iy \end{aligned}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

$\xi$ a real n-plane bundle

$\xi \underset{\mathbb{R}}{\otimes} \mathbb{C}$ a complex n-plane bundle

$$\xi \underset{\mathbb{R}}{\otimes} \mathbb{C} \cong_{\mathbb{C}} \overline{\xi \underset{\mathbb{R}}{\otimes} \mathbb{C}}$$

$\Rightarrow$ $\forall k$

$$c_k(\xi \otimes \mathbb{C}) = c_k(\overline{\xi \otimes \mathbb{C}}) = (-1)^k c_k(\xi \otimes \mathbb{C})$$

$\Rightarrow$ $\forall k = 2i+1$, $2c_k(\xi \otimes \mathbb{C}) = 0$. (ignored).

Def: The i-th Pontrjagin class of a real vector bundle $\xi$ over $B$ is

$$p_i(\xi) := (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B;\mathbb{Z})$$

note: $p_i(\xi) = 0 \quad \forall i > \frac{n}{2}$. $n = \dim_{\mathbb{R}} \xi$.

$$p(\xi) := 1 + p_1(\xi) + \cdots + p_{[\frac{n}{2}]}(\xi)$$

Say: $p$ inherits properties of $c$, e.g.

$$p(\xi \oplus \eta) = p(\xi)\, p(\eta) \quad + \quad (\text{2-torsions}).$$

i.e. $$2\,(p(\xi \oplus \eta) - p(\xi)\, p(\eta)) = 0.$$

$\boxed{\text{STOP}}$

Say: Compare $w_i$ and $p_i$

$w_i$ is 2-torsion.

$p_i$ ignores 2-torsions. So $p_i$ 'detects' info about $\xi$ that $w_i$ ignores.

Ex:
$$\tau_{S^n} \oplus \underset{\varepsilon}{\underset{\shortparallel}{\nu^{\mathbb{R}^{n+1}}_{S^n}}} = \tau_{\mathbb{R}^{n+1}}\big|_{S^n} = \varepsilon^{n+1}$$

$$\Rightarrow \quad p(\tau_{S^n}) = p(\tau_{S^n} \oplus \varepsilon) = p(\varepsilon^{n+1}) = 1.$$

**Rmk:** Equivalent definition using $BG$:

complexification: $O(n) \longrightarrow U(n)$ (say Midterm $n=3$)

$A \longmapsto A \otimes_{\mathbb{R}} \mathbb{C}$

$\rightsquigarrow \quad BO(n) \longrightarrow BU(n).$

$\rightsquigarrow \quad H^{4i}(BO(n);\mathbb{Z}) \longleftarrow H^{4i}(BU(n);\mathbb{Z})$

$(-1)^i p_i \longleftarrow\!\shortmid \; c_{2i}$

$$\begin{tikzcd}
& \text{real} \arrow[dr, "\otimes\mathbb{C}"] & \\
\text{oriented} \arrow[ur, bend left, "\text{forget orientation.}"] & & \text{complex} \arrow[ll, bend left, "\text{forget complex}"]
\end{tikzcd}$$

Say: Going around, we get a new bundle of the same type with twice dim.

What is the relationship ?

① Start with real: $\xi \overset{\text{around}}{\rightsquigarrow} (\xi\otimes\mathbb{C})_{\mathbb{R}}$

$$(\xi\otimes\mathbb{C})_{\mathbb{R}} \cong \xi\oplus\xi .$$

② Start with complex: $\omega$ a complex vector bundle $\dim_{\mathbb{C}} = n$

$$\omega \overset{\text{around}}{\rightsquigarrow} \omega_{\mathbb{R}}\otimes\mathbb{C}$$

$$\underset{\underset{\dim_{\mathbb{C}}=2n}{\uparrow}}{\omega_{\mathbb{R}}\otimes\mathbb{C}} \cong \omega\oplus\bar{\omega} .$$

Pf (you): On each fiber, the map $\omega_{\mathbb{R}}\otimes\mathbb{C} \xrightarrow{g} \omega\oplus\bar{\omega}$

$$x \longmapsto (x, -ix)$$

is $\mathbb{C}$-linear. iso.

~~$g(i\cdot x) = (ix, (-i)ix) = (ix, x)$~~

~~$J(g(x)) = J(x, -ix) = (ix, (-i)\cdot -ix) = (ix, x)$~~

Here, let $p_k := p_k(\omega_{\mathbb{R}})$

$c_i := c_i(\omega)$.

we have

$$\underbrace{1-p_1+p_2-\cdots\pm p_n}_{\substack{\| \\ c(\omega\oplus\bar{\omega}) \\ \| \\ c(\omega_{\mathbb{R}}\otimes\mathbb{C}) \\ \| \\ p(\omega_{\mathbb{R}})}} = (\underbrace{1+c_1+\cdots+c_n}_{\substack{\| \\ c(\omega)}})\cdot(\underbrace{1-c_1+c_2\cdots\pm c_n}_{\substack{\| \\ c(\bar{\omega})}})$$

(after degree shift and signs)

<u>Cor</u>: For complex $\omega$,

Chern classes of $\omega$ determine Pontrjagin classes of $\omega_{\mathbb{R}}$ by

$$p_k(\omega_{\mathbb{R}}) = c_k^2 - 2c_{k-1}c_{k+1} + \cdots \cancel{\mp} \pm 2c_1c_{2k-1} \mp 2c_{2k}$$

$c_i = c_i(\omega)$.

(*)

**Rmk**: (via $BG$).

$$\begin{tikzcd}
 & O(2n) \arrow[rd, bend left] & \\
SO(2n) \arrow[ru, bend left] & U(n) \arrow[l, bend left] \arrow[r, dashed] & U(2n)
\end{tikzcd}$$

(go around)

$$U(n) \to SO(2n) \longrightarrow O(2n) \longrightarrow U(2n).$$

$$H^\bullet(BU(n)) \leftarrow H^\bullet(BO(2n)) \leftarrow H^\bullet(BU(2n))$$

$$\text{RHS of } (*). \quad \underset{\text{last result}}{\longmapsfrom} \quad p_i \quad \underset{\text{def. of } p_i}{\longmapsfrom} \quad (-1)^i c_{2i}$$

Example: $\tau := \tau_{\mathbb{C}P^n}$. Goal: $p_k(\tau_{\mathbb{C}P^n}) = ?$

before: $c(\tau) = (1+a)^{n+1}$ $\quad a \in H^2(\mathbb{C}P^n; \mathbb{Z})$

$$\left[\begin{array}{l} \tau_{\mathbb{C}P^n} \cong \mathrm{Hom}(\gamma^1, (\gamma^1)^\perp) \\ \tau \oplus \varepsilon^1 \cong \mathrm{Hom}(\gamma^1, (\gamma^1)^\perp \oplus \gamma^1) = \bigoplus_{n+1} (\gamma^1)^\vee \end{array}\right]$$

$a = -c_1(\gamma^1_n)$ = hyperplane class.

$p_k := p_k(\tau_{\mathbb{R}})$.

$$1 - p_1 + \cdots \pm p_n = c(\tau)\, c(\bar{\tau}) = (1+a)^{n+1}(1-a)^{n+1}$$
$$= (1-a^2)^{n+1}.$$

$\Rightarrow 1 + p_1 + \cdots + p_n = (1+a^2)^{n+1}$.

$$\Rightarrow \quad p_k = \binom{n+1}{k} a^{2k} \qquad \forall\ 1 \le k \le \frac{n}{2}.$$

Hence, each $p_k$ can be nonzero.

③ Start with oriented. $\quad \xi$ real. oriented. dim $= n$.

$$\xi \overset{\text{around}}{\rightsquigarrow} (\xi \otimes \mathbb{C})_{\mathbb{R}}$$

We know

$$(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi \qquad \text{as in ①}$$

$(v_1, \cdots, v_n)$ positively oriented basis for $\xi_b$.

$(v_1, iv_1, \cdots, v_n, iv_n)$ - - - - for $(\xi \otimes \mathbb{C})_{\mathbb{R}}$

$\Big\downarrow$ sign changes $\binom{n}{2}$ - times.

$(v_1, \cdots, v_n, v_1, \cdots, v_n)$ - - - - for $\xi \oplus \xi$.

Prop. $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$

with orientation preserved iff $\binom{n}{2}$ is even.

Cor: $\xi$ real $2n$-dim oriented,

$$\Rightarrow \quad p_n(\xi) = e(\xi)^2 \in H^{4n}(B;\mathbb{Z}).$$

Pf: $p_n(\xi) = (-1)^n c_{2n}(\xi \otimes \mathbb{C}) = (-1)^n e(\xi \overset{''}{\otimes} \mathbb{C})$

$= (-1)^n e(\xi \oplus \xi) \cdot (-1)^{\binom{2n}{2}} \rightsquigarrow = (-1)^n = e(\xi) \cdot e(\xi)$ $\square$

**Rmk:** (via BG):

$$\begin{tikzcd} & O(n) \arrow[dr] & \\ SO(n) \arrow[ur] & & U(n) \arrow[dl, dotted] \\ & {} & \end{tikzcd}$$

$$\begin{tikzcd} 2n & n \arrow[l, mapsto] \end{tikzcd}$$

$$\begin{tikzcd} SO(n) \arrow[r] & O(n) \arrow[r] \arrow[d, squiggly] & U(n) \arrow[r] & SO(2n) \\ H^{2n}BSO(n) & H^{2n}BO(n) \arrow[l] & H^{2n}BU(n) \arrow[l] & H^{2n}BSO(2n) \arrow[l] \\ e^2 & & & p_n(\tilde{\gamma}^{2n}) \arrow[lll, mapsto] \end{tikzcd}$$

$$\begin{tikzcd}
\tilde{\gamma}_n \arrow[d] \\
BSO(n) = \tilde{G}_n
\end{tikzcd}$$

**Thm**: Let $R$ be an integral domain s.t. $\frac{1}{2} \in R$.
(e.g. $R = \mathbb{Q}, \mathbb{Z}[\frac{1}{2}]$).

$$H^\bullet(\tilde{G}_{2m+1}; R) \cong R[p_1, \cdots, p_m]$$

$$H^\bullet(\tilde{G}_{2m}; R) \cong R[p_1, \cdots, p_{m-1}, e]$$

where $p_i = p_i(\tilde{\gamma}^n)$. $e = e(\gamma^n)$.

Equivalently.

$$H^\bullet(\tilde{G}_n; R) = R[p_1, \cdots, p_{[\frac{n}{2}]}, e] \Big/ \left( \begin{array}{ll} e = 0 & n \text{ odd} \\ e = p_{\frac{n}{2}}^2 & n \text{ even} \end{array} \right).$$

**pf sketch:** Induction on $n$.

$n=1$. $\tilde{G}_1 = S^\infty \simeq *$. $( BSO(1) = B* )$.

For induction, recall $H \subseteq G$

$$\Rightarrow \quad G/H \to BH \to BG$$

fibration.

$$\begin{tikzcd} \frac{SO(n+1)}{SO(n)} \arrow[r] \arrow[d, phantom, "\parallel"] & BSO(n) \arrow[r] \arrow[d, phantom, "\parallel"] & BSO(n+1) \arrow[d, phantom, "\parallel"] \\ S^n \arrow[r] & \tilde{G}_n \arrow[r] \arrow[d, phantom, "\text{is}"] & \tilde{G}_{n+1} \\ & E_0(\tilde{\gamma}_{n+1}) & \end{tikzcd}$$

oriented bundle

Gysin sequence:

$$\cdots \to H^i(\tilde{G}_{n+1}) \xrightarrow{\cup e} H^{i+n}(\tilde{G}_{n+1}) \to H^{i+n}(\tilde{G}_n) \to H^{i+1}(\tilde{G}_{n+1})$$

$e = e(\tilde{\gamma}_{n+1})$.

Use induction hypothesis. ...

**Note:** $n$ even $\Rightarrow 2e(\tilde{\gamma}_{n+1}) = 0 \overset{\frac{1}{2} \in \mathbb{R}}{\Longrightarrow} e(\tilde{\gamma}_{n+1}) = 0$

Week 12

Wednesday, November 30, 2022

## Pontrjagin classes.

Last time: $\xi$ = real n-plane bundle.

$\xi \otimes_{\mathbb{R}} \mathbb{C}$ = complex n-plane bundle $\quad (\xi \otimes \mathbb{C} \cong \overline{\xi \otimes \mathbb{C}})$

Def: $p_i(\xi) := (-1)^i c_{2i}(\xi \otimes \mathbb{C}) \in H^{4i}(B;\mathbb{Z})$

$p(\xi) := 1 + p_1(\xi) + \cdots + p_{\lfloor \frac{n}{2} \rfloor}(\xi)$.

$p(\xi \oplus \eta) = p(\xi)\, p(\eta)$ + 2 torsions.

Example. $\tau_{S^n}$. $\quad S^n \subseteq \mathbb{R}^{n+1}$

$\Rightarrow \tau_{S^n} \oplus \underbrace{\nu_{S^n}}_{\overset{\shortparallel}{\varepsilon^1}} \cong \varepsilon^{n+1}$

$\Rightarrow \tau_{S^n} \oplus \varepsilon^1 \cong \varepsilon^{n+1}$

$p(\tau_{S^n}) = p(\tau_{S^n} \oplus \varepsilon^1) = p(\varepsilon^{n+1}) = 1$

+ 2 torsions. $\downarrow 0$ $\quad H^*(S^n)$ is free $\mathbb{Z}$-modules

Rmk: Equivalent definition using BG

complexification: $O(n) \hookrightarrow U(n)$

$A \longmapsto A$

$\rightsquigarrow BO(n) \longrightarrow BU(n)$

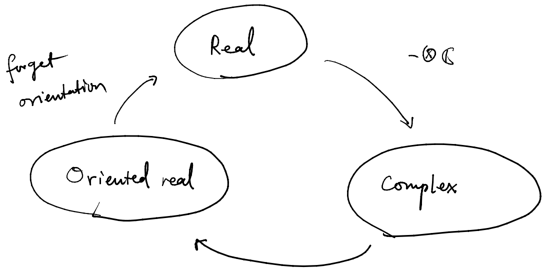
$\rightsquigarrow H^{4i}(BO(n);\mathbb{Z}) \longleftarrow H^{4i}(BU(n);\mathbb{Z})$

image $= (-1)^i p_i \longleftarrow\!\shortmid\, c_{2i}$

Rmk: n=3.

$SO(3) \to SU(3)$

appears in midterm



① Start with a real $\xi$, go around: $\xi \longmapsto (\xi \otimes \mathbb{C})_{\mathbb{R}}$

$(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$ as real vector bundles.

② Start with complex $\omega$, $\dim_{\mathbb{C}} \omega = n$

$\omega \xrightarrow{\text{go around}} \omega_{\mathbb{R}} \otimes \mathbb{C}$

claim: $\omega_{\mathbb{R}} \otimes \mathbb{C} \cong \omega \oplus \overline{\omega}$ as complex vector bundles.

Pf: On each fiber, the map $\omega_{\mathbb{R}} \otimes \mathbb{C} \longrightarrow \omega \oplus \overline{\omega}$, $x \longmapsto (x, -ix)$

(you): is a $\mathbb{C}$-linear isomorphism.

Hence, we have

$$c(\omega)\, c(\overline{\omega}) = c(\omega \oplus \overline{\omega}) = c(\omega_{\mathbb{R}} \otimes \mathbb{C})$$

let $c_i := c_i(\omega)$

$p_i := p_i(\omega_{\mathbb{R}})$

$$\text{LHS} = (1 + c_1 + c_2 + \cdots c_n)(1 - c_1 + c_2 - \cdots \pm c_n)$$

$$\overset{\star}{\shortparallel}$$

$$\text{RHS} = p^{\pm}(\omega_{\mathbb{R}}) = 1 - p_1 + p_2 - \cdots \pm p_n.$$

( after sign changes) ~~(after~~

Cor: For complex $\omega$, chern classes of $\omega$ determine Pontryagin classes of $\omega_{\mathbb{R}}$

$$p_k(\omega_{\mathbb{R}}) = \underbrace{c_k^2 - 2c_{k-1}c_{k+1} + \cdots \pm 2c_1 c_{2k-1} \mp 2c_{2k}}_{\text{where } c_i := c_i(\omega)}$$

Rmk: (via BG)

$$\begin{tikzcd}
 & & O(2n) \arrow[dr, bend left] & \\
SO(2n) \arrow[urr, bend left] & U(n) \arrow[l, bend left] \arrow[rr, dashed] & & U(2n)
\end{tikzcd}$$

$$\begin{tikzcd}
U(n) \arrow[r] & SO(2n) \arrow[r] & O(2n) \arrow[r] & U(2n)
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(BU(n)) & & H^*(BO(2n)) \arrow[ll] & H^*(BU(2n)) \arrow[l] \\
(\text{RHS of Cor.}) & & p_i \arrow[ll, mapsto, "\text{Cor}"'] & (-1)^i c_{2i} \arrow[l, mapsto, "\text{def}"]
\end{tikzcd}$$

Example: $\tau := \tau_{\mathbb{C}P^n}$ Goal: $p_k(\tau_{\mathbb{C}P^n}) = ?$

$$\overset{p_k((\tau_{\mathbb{C}P^n})_{\mathbb{R}})}{\shortparallel}$$

Recall: $c(\tau) = (1+a)^{n+1}$ where $a = -c_1(\gamma_n^1) = $ hyperplane class $\in H^2(\mathbb{C}P^n)$.

$$\overset{c_1((\gamma_n^1)^\vee)}{\shortparallel}$$

$$\left[\begin{aligned}
&\tau_{\mathbb{C}P^n} \cong \text{Hom}(\gamma^1, (\gamma^1)^\perp) \\
&\Rightarrow \tau_{\mathbb{C}P^n} \oplus \varepsilon^1 \cong \text{Hom}(\gamma^1, (\gamma^1)^\perp) \oplus \underbrace{\text{Hom}(\gamma^1, \gamma^1)}_{\overset{\shortparallel}{\varepsilon^1}} \cong \text{Hom}(\gamma^1, \underbrace{(\gamma^1)^\perp \oplus \gamma^1}_{\varepsilon^{n+1}}) \\
&\qquad = (\text{Hom}(\gamma^1, \varepsilon^1))^{\oplus(n+1)} \\
&\Rightarrow c(\tau) = c(\tau \oplus \varepsilon^1) = c((\gamma^1)^\vee)^{n+1} = (1+a)^{n+1}.
\end{aligned}\right]$$

Set $p_k := p_k(\tau_{\mathbb{R}})$

Before $\Rightarrow$ $1 - p_1 + p_2 - \cdots \pm p_n = c(\tau)\, c(\overline{\tau}) = (1+a)^{n+1}(1-a)^{n+1}$

$$= (1-a^2)^{n+1}$$

$\Rightarrow 1 + p_1 + p_2 + \cdots + p_n = (1+a^2)^{n+1}$. $(\star)$

$$p_k = \binom{n+1}{k} a^{2k} \neq 0$$

$p_n - (1+a^2)$ . (☆)

$$p_k = \binom{n+1}{k} a^{2k} \neq 0 \qquad \forall\ 1 \le k \le \frac{n}{2}$$
in $H^{4k}(\mathbb{C}P^n; \mathbb{Z})$

③ Start with an oriented $\xi$ $\dim_{\mathbb{R}} = n$.

$$\xi \xrightarrow{\text{go around}} (\xi \otimes \mathbb{C})_{\mathbb{R}}$$

We knew from ① that $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$
$\uparrow$ $\uparrow$

orientation ?

take $(v_1, \cdots, v_n)$ positively oriented basis for $\xi_b$ $b \in B$.

$(v_1, iv_1, v_2, iv_2, \cdots, v_n, iv_n)$ is a positively oriented basis for $(\xi \otimes \mathbb{C})_{\mathbb{R}}$

$\Big\downarrow$ sign changes $\binom{n}{2}$-times.

$(v_1, \cdots, v_n, v_1, \cdots, v_n)$ is a — — — — — — — for $\xi \oplus \xi$

Prop. $(\xi \otimes \mathbb{C})_{\mathbb{R}} \cong \xi \oplus \xi$ with orientation preserved iff $\binom{n}{2}$ is even.

Cor. $\xi$ real $2n$-dim oriented,

$$\Rightarrow p_n(\xi) = e(\xi)^2 \in H^{4n}(B; \mathbb{Z})$$

pf. $p_n(\xi) \underset{\text{def}}{=} (-1)^n c_{2n}(\xi \otimes \mathbb{C}) = (-1)^n e(\xi \otimes \mathbb{C})$

$$\overset{\text{Cor}}{=} (-1)^n e(\xi \oplus \xi)(-1)^{\binom{2n}{2}}$$

$$= e(\xi \oplus \xi) = e(\xi) \cdot e(\xi)$$ □.

Rmk (via $BG$)

$$\begin{tikzcd} & O(n) \arrow[rrd] & & \\ SO(2n) & SO(n) \arrow[l, dashed] \arrow[u, bend left] & & U(n) \arrow[lll, bend left] \end{tikzcd}$$

$$SO(n) \to O(n) \longrightarrow U(n) \longrightarrow SO(2n)$$

$$H^{\cdot} BSO(n) \leftarrow \cdots \quad - - - - \leftarrow H^{\cdot} BSO(2n)$$

$$e^2 \longleftarrow\!\!\!\!\longleftarrow\!\!\!\shortmid \; p_n$$

Universal bundles:

$$\begin{tikzcd} \{\pm 1\} \cong \frac{O(n)}{SO(n)} \arrow[r] & BSO(n) \arrow[r, "2:1"] \arrow[d, phantom, "\wr"] & BO(n) \arrow[d, phantom, "\shortparallel"] \\ \{\pm 1\} \arrow[r] & \widetilde{G}_n \arrow[r, "2:1"] & G_n \end{tikzcd}$$

$$\widetilde{G}_n = \{ X \mid X \le \mathbb{R}^\infty, \text{ oriented}, \dim = n \}$$

Thm. Let $R$ be an integral domain s.t. $\frac{1}{2} \in R$ (e.g. $R = \mathbb{Q}$, $R = \mathbb{Z}[\frac{1}{2}]$)

$$H^{\cdot}(\widetilde{G}_{2m+1}; R) \cong R[p_1, \cdots, p_m]$$

**Thm.** Let $R$ be an integral domain s.t. $\frac{1}{2} \in R$ (e.g. $R = \mathbb{Q}$, $R = \mathbb{Z}[\frac{1}{2}]$)

$$H^\bullet(\widetilde{G}_{2m+1} ; R) \cong R[p_1, \dots, p_m]$$

$$H^\bullet(\widetilde{G}_{2m} : R) \cong R[p_1, \dots, p_{m-1}, e]$$

where $p_i = p_i(\tilde{\gamma}^n)$, $e = e(\tilde{\gamma}^n)$

Equivalently,

$$H^\bullet(\widetilde{G}_n ; R) = R[p_1, \dots, p_{\lfloor \frac{n}{2} \rfloor}, e] \Big/ \left( \begin{cases} e = 0 & n \text{ odd} \\ \cancel{e = p^2_{\frac{n}{2}}} & n \text{ even.} \end{cases} \right)$$

$e^2 = p_{\frac{n}{2}}$

pf skipped.

**Cor:** For $R$ as above,

$$H^\bullet(G_n ; R) = R[p_1, p_2, \dots, p_{\lfloor \frac{n}{2} \rfloor}]$$

## Applications.

preview: we will prove:

- $\nexists$ orientation-reversing diffeo. $\mathbb{CP}^{2n} \longrightarrow \mathbb{CP}^{2n}$.
- → $\mathbb{CP}^{2n} \neq \partial V^{2n+1}$
- $\mathbb{CP}^{2n} \ncong X \times Y$ where $X$ or $Y \neq$ a pt.

Tool: Chern and Pontrjagin numbers.

**Def.** A partition of $n$ is an unordered sequence of positive integers $I = (i_1, \dots, i_r)$ s.t. $i_1 + \dots + i_r = n$.

Write $I \vdash n$.

$I = (i_1, \dots i_r)$ $\quad I \vdash n$

$J = (j_1, \dots, j_s)$ $\quad J \vdash m$

$IJ := (i_1, \dots, i_r, j_1, \dots, j_s)$ $\quad IJ \vdash n+m$.

$\uparrow$ "juxtaposition"

**Def:** A refinement of $I = (i_1, \dots, i_r)$ is a partition

$I' = I_1 I_2 \cdots I_r$ s.t. $I_j \vdash i_j$ $\quad \forall j \leq r$.

Partition number $p(n) :=$ # of partitions of $n$.

Chern numbers.: $K^n =$ closed complex manifold $\dim_{\mathbb{C}} = n$.

$I = (i_1, \dots, i_r)$, $I \vdash n$

**Def:** The $I$-th Chern number of $K^n$ is

$$c_I[K^n] := \langle \underbrace{c_{i_1}\cdots c_{i_r}}_{\substack{\in \\ H^{2n}(K;\mathbb{Z})}},\ \underbrace{[K^n]}_{\substack{\in \\ H_{2n}(K;\mathbb{Z})}} \rangle_K \in \mathbb{Z}$$

$c_i = c_i(\tau_K)$

Equivalently, let $f$ be the classifying map of $\tau_K$

$$f : K \longrightarrow G_n(\mathbb{C}^\infty) \quad \text{s.t. } f^*\gamma^n = \tau_K.$$

$f_*[K] \in H_{2n}(G_n;\mathbb{Z})$

$$c_I[K^n] = \langle \underset{\substack{\in \\ H^{2n}(G_n)}}{c_{i_1}\cdots c_{i_r}} : \underset{\substack{\in \\ H_{2n}(G_n)}}{f_*[K]} \rangle_{G_n}$$

Since $H^*(G_n;\mathbb{Z}) = \mathbb{Z}[c_1, \dots, c_n]$

$\{c_I : I \vdash n\}$ forms a $\mathbb{Z}$-basis for $\underset{\text{rank} = p(n)}{H^{2n}(G_n;\mathbb{Z})}$.

$\Rightarrow$ Chern numbers $\{c_I[K] : I \vdash n\} \subseteq \mathbb{Z}$ completely determine $f_*[K] \in H_{2n}(G_n;\mathbb{Z})$.

Pontrjagin numbers. $M^{4n}$ := closed, oriented manifold, $\dim_{\mathbb{R}} = 4n$.

$I = (i_1 \cdots i_r)$, $I \vdash n$

Def: The I-th Pontrjagin number of $M^{4n}$ is

$$P_I[M^{4n}] = \langle \underbrace{\underset{\substack{\in \\ H^{4n}(M;\mathbb{Z})}}{p_{i_1}\cdots p_{i_r}},\ \underset{\substack{\in \\ H_{4n}(M;\mathbb{Z})}}{[M^{4n}]}} \rangle \in \mathbb{Z}$$

Rmk: $\bar{M}^{4n}$ := $M^{4n}$ with opposite orientation

$[\bar{M}] = -[M]$

but $p_k(\tau_M) = p_k(\tau_{\bar{M}}) \in H^{4k}(M;\mathbb{Z})$

$\Rightarrow$ $P_I[\bar{M}] = -P_I[M]$.

Cor: If $M^{4n}$ has some nonzero Pontrjagin number, then $\nexists$ orientation-reversing diffeo $M \to M$.

Ex: ($M = \mathbb{C}P^m$)

Before: $p(\tau_{\mathbb{C}P^m}) = (1+a^2)^{m+1}$ where $a$ = hyperplane class $= -c_1(\gamma^1) \in H^2$

$p_k(\mathbb{C}P^m) = \underset{\leftharpoonup}{\binom{m+1}{k}} a^{2k} \in H^{4k}$.

$\boxed{\text{Take } m = 2n}$ $M^{4n} = \mathbb{C}P^{2n}$

$\boxed{\text{Take } m=2n}$ $M^{4n} = \mathbb{C}P^{2n}$

$\forall I \vdash n$.

$$\boxed{p_{i_1}\cdots p_{i_r}[\mathbb{C}P^{2n}] = \binom{2n+1}{i_r}\cdots\binom{2n+1}{i_r} \neq 0}$$

Cor

$\Rightarrow$ $\nexists$ orientation-reversing diffeo $\mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$

In contrast, complex conjugation $\mathbb{C}P^{2n-1} \to \mathbb{C}P^{2n-1}$ is an orientation-reversing diffeo.

note: $\mathbb{C}P^{2n-1}$ has no top Pontrjagin classes.

since $[\mathbb{C}P^{2n-1}] \in H_{4n-2} \neq H_{4k}$.

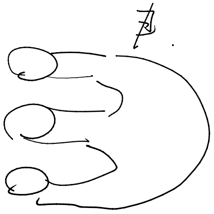
prop. If $M^{4n} = \partial V^{4n+1}$ where $V$ is compact, oriented, then all Pontrjagin numbers of $M^{4n} = 0$.

(HW).


✓

Cor. $\mathbb{C}P^{2n} \neq \partial V^{4n+1}$ ↗ oriented

Moreover, $\underbrace{\mathbb{C}P^{2n} \sqcup \mathbb{C}P^{2n} \sqcup \cdots \sqcup \mathbb{C}P^{2n}}_{\text{any finite number}} \neq \partial V$



note: This fails for $\mathbb{C}P^{2n+1}$ e.g. $\mathbb{C}P^1 \cong S^2 = \partial D^3$

## Symmetric function theory

Recall:

$$\begin{tikzcd} T^n = U(1)^{\times n} \arrow[r, hook] & U(n) \\ \lambda_1\cdots\lambda_n \arrow[r, mapsto] & \begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix} \end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd} BT^n \arrow[r] & BU(n) \end{tikzcd}$$

$$\rightsquigarrow$$

$$\begin{tikzcd} H^*(BT^n;\mathbb{Z}) & H^*(BU(n);\mathbb{Z}) \arrow[l] & \\ S_n \circlearrowright \mathbb{Z}[t_1,\dots,t_n] \arrow[u, equal] & \mathbb{Z}[c_1,\dots,c_n] \arrow[u, equal] \arrow[l] \arrow[d] & c_i \arrow[d, mapsto] \\ \boxed{\mathbb{Z}[t_1,\dots,t_n]^{S_n}} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\cong"] & \mathbb{Z}[\sigma_1,\dots,\sigma_n] & \sigma_i(t_1,\dots,t_n) \end{tikzcd}$$

FTSP

$|t_i| = 1$

$|\sigma_i| = i$

$|t_i| = 1$
$|\sigma_i| = i$

$$\underbrace{\boxed{\mathbb{Z}[t_1, \dots, t_n]^{S_n}} \cong \mathbb{Z}[\sigma_1, \dots, \sigma_n]}_{\text{FTSP}} \quad \overset{\Downarrow}{\sigma_i}(t_1, \dots, t_n)$$

where $1 + \sigma_1 + \cdots + \sigma_n \overset{\text{def.}}{=} (1+t_1)(1+t_2)\cdots(1+t_n)$

Next, we will use / introduce general results about symmetric poly. (algebra, combinatorics) in $\mathbb{Z}[t_1, \dots t_n]^{S_n}$.

$$\shortparallel$$
$$\mathbb{Z}[\sigma_1, \dots \sigma_n]$$

Let $S := \mathbb{Z}[\sigma_1, \dots, \sigma_n] = \mathbb{Z}[t_1, \dots, t_n]^{S_n} = \bigoplus_{k=0}^{\infty} S^k$

$S^k_{(n)} = \{$ symmetric polynomials with $\deg = k \}$ (in $n$-variables)

A $\mathbb{Z}$-basis for $S^k$ is :

$$\{ \underbrace{\sigma_{i_1} \cdots \sigma_{i_r}}_{\text{def. } \sigma_I} \mid \underbrace{i_1 + i_2 + \cdots + i_r}_{I \text{ partition of } k} = k, \text{ each } i_j \le n \}$$

$S^k = \mathbb{Z}\{ \sigma_I \mid I \vdash k, \text{ each } i_j \text{ in } I \text{ is } \le n \}$. ⟵ a basis

Two monomials are equivalent if they differ by a permutation.

e.g. $t_1 t_2^3 t_3^5 \sim t_4 t_7^3 t_2^5$

For $I \vdash k$, the $I$-th monomial symmetric polynomial is

$$m_I := \sum \text{monomials equivalent to } t^I = t_1^{i_1} t_2^{i_2} \cdots t_r^{i_r}$$

eg. $k = 3$, $I = 2+1 = 3$

$$m_I = t_1 t_2^2 + t_1^2 t_2 + t_1 t_3^2 + t_1^2 t_3 + \cdots$$

Lemma : $\{ m_I \mid I \vdash k, \ r \le n \}$ forms a $\mathbb{Z}$-basis for $S^k = S^k_{(n)}$.

Pf. They clearly span. Now check dimensions.

$\dim_{\mathbb{Z}} S^k_{(n)} = \# \{\sigma_I\}\text{'s} = \{ I \mid I \vdash k, \text{ each } i_j \le n \}$ ... ①

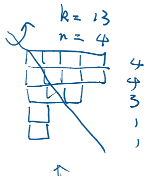
want ↳ $= \# \{m_I\}\text{'s} = \{ I \mid I \vdash k, \text{ length } r \le n \}$ --- ②

① $= \# \{$ Young diagrams with $k$ boxes with $\le n$ columns $\}$

$\updownarrow$

② $= \# \{$ Young diagrams $\}$

eg. $I = (4,4,3,1,1)$ Young diagram



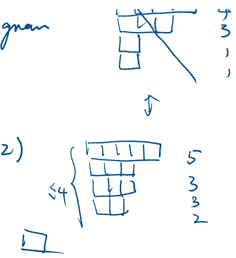
…, $\leq n$ columns ) Young diagram

$\updownarrow$

② $= \# \left\{ \begin{array}{l} \text{Young diagrams} \\ \text{with } k \text{ boxes} \\ \text{and } \leq n \text{ rows} \end{array} \right\}$

e.g. $I = (5,3,3,2)$

Young



$k=13$
$n=4$

For any $I \vdash k$,

$$m_I(t_1, \cdots, t_n) \in \mathbb{Z}[t_1 \cdots t_n]^{S_n} \overset{FTSP}{\cong} \mathbb{Z}[\sigma_1 \cdots \sigma_n]$$

$$\Rightarrow \boxed{m_I(t_1 \cdots t_n) = S_I(\sigma_1, \cdots, \sigma_n)} \qquad \sigma_i = \sigma_i(t_1 \cdots t_n)$$

$S_I$ is a polynomial in $n$-variables.

Moreover, one checks $S_I$ is the same $\forall n \geq k$.

and $\forall n < k$, $\quad S_{I,n}(\sigma_1, \cdots, \sigma_n) = S_{I,k}(\sigma_1, \cdots, \sigma_n, 0, \cdots, 0)$

So $S_I$ is independent of $n$.

$$\begin{array}{ccc} S_{I,n}(\sigma_1, \cdots, \sigma_n) & = & S_{I,k}(\sigma_1, \cdots, \sigma_n, 0, \cdots, 0) \\ \| & & \| \\ m_I(t_1 \cdots t_n) & & m_I(t_1 \cdots t_n\ t_{n+1} \cdots t_k) \end{array}$$

In summary, if $n \geq k$,

then $\{ m_I \mid I \vdash k \} = \{ S_I(\sigma_1, \cdots, \sigma_n) \mid I \vdash k \}$

forms a $\mathbb{Z}$-basis for $S^k = \mathbb{Z}[\sigma_1 \cdots \sigma_n] \big|_{\deg = k}$.

e.g. $k=1$, $\quad I \vdash 1 : 1=1$. $\quad S_1(\sigma_1) = \sigma_1$

$k=2$, $\quad S_2(\sigma_1, \sigma_2) = \sigma_1^2 - 2\sigma_2 = (t_1+t_2)^2 - 2t_1t_2 = \underbrace{t_1^2 + t_2^2}_{\overset{\|}{m_2(t_1, t_2)}}$

$S_{1,1}(\sigma_1, \sigma_2) = t_1 t_2 = \sigma_2$.

$k=n=2$.
$\sigma_1 = t_1 + t_2$
$\sigma_2 = t_1 t_2$
$m_2 = t_1^2 + t_2^2$

$k=3$, $\quad S_3(\sigma_1\ \sigma_2\ \sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$

$S_{1,2}(\quad) = \sigma_1\sigma_2 - 3\sigma_3$

$S_{1,1,1} = \sigma_3$.

<u>Apply to topology</u>. $\quad H^*(BU(n)) \longrightarrow H^*(BU(1)^{\times n})^{S_n}$

$$c_i \longmapsto \sigma_i(t_1, \cdots, t_n)$$

$\{ c_I \mid I \vdash k,\ i_j \leq n \}$ forms a $\mathbb{Z}$-basis for $H^{2k}(BU(n))$

$\Big\downarrow$ change of basis

$\{ S_I(c_1, \cdots, c_n) \mid I \vdash k,\ r \leq n \}$ . . ——— - — -

<u>Def</u>: For any complex vector bundle $V^{\omega}$ of dim $n$

for any partition $I \vdash k$,

define

$$S_I(c(\omega)) := S_I(c_1(\omega), \cdots, c_n(\omega)) \in H^{2k}(B; \mathbb{Z})$$

<u>Lemma</u> (Thom)

$$S_I(c(\omega \oplus \omega')) = \sum_{\substack{JK=I \\ \nwarrow \text{ juxtaposition}}} S_J(c(\omega))\, S_K(c(\omega'))$$

Pf: Pure algebra.

Want: If $\sigma_k$ = $k$-th elementary symmetric poly in $t_1 \cdots t_n$

$\sigma_k' :=$ - - - - - - - - - - - - - - in $t_{n+1} \cdots t_{n+m}$

$\sigma_k'' := \sum_{i=0}^{k} \sigma_i \sigma_{k-i}'$

check: $\sigma_k''$ = - —— - - - - - - - - - in $t_1, \cdots, t_{n+m}$.

then $S_I(\sigma_1'', \cdots, \sigma_k'') = \sum_{JK=I} S_J(\sigma_1 \cdots) S_K(\sigma_1' \cdots)$

pf LHS $\overset{\text{def}}{=} m_I(t_1 \cdots t_{n+m}) = \sum t_{a_1}^{i_1} \cdots t_{a_r}^{i_r}$, $a_1 \cdots a_r$ distinct in $\{1, \cdots, n+m\}$

let $J := \{ i_q : 1 \le a_q \le n \}$

$K := \{ i_q : n+1 \le a_q \le n+m \}$ $\quad JK = I.$

$\square.$

When $I \vdash k$ is $\underset{\sim}{k=k}$, we write $S_k = S_I$.

The only $J, K$ s.t. $JK = I = k$ is $J = \phi$ or $K = \phi$.

Cor. $S_k(c(\omega \oplus \omega')) = S_k(c(\omega)) + S_k(c(\omega'))$

$S_k$ takes sum to sum. (unlike $c_k$)

Rmk. One can define formally "the Chern character" of $\omega$

$$ch(\omega) := n + \sum_{k=1}^{\infty} \frac{S_k(c(\omega))}{k!} \in H^{\Pi}(B;\mathbb{Q})$$

Then $ch(\omega \oplus \omega') = ch(\omega) + ch(\omega')$

$ch(\omega \otimes \omega') = ch(\omega) \cdot ch(\omega')$

Compare to total chern class $c(\omega) \in H^{\Pi}(B;\mathbb{Z})$

$c(\omega \oplus \omega') = c(\omega)c(\omega')$

$c(\omega \otimes \omega') = \underset{\substack{\uparrow \\ \text{some polynomials. complicated.}}}{P}(c(\omega), c(\omega'))$

Now take $K^n :=$ complex manifold,

for each $I \vdash n$, consider the characteristic numbers.

$$S_I[K^n] := \langle S_I(c(\tau_K)), [K^n] \rangle \in \mathbb{Z}.$$

Cor. (Product formula for $S_I$):

For $I \vdash (m+n)$

$$S_I[K^m \times L^n] = \sum_{I_1 I_2 = I} S_{I_1}[K^m]\, S_{I_2}[K^n]$$

$$S_I[K^m \times L^n] = \sum_{\substack{I_1 I_2 = I \\ I_1 \vdash m,\ I_2 \vdash n}} S_{I_1}[K^m]\ S_{I_2}[K^n]$$

证. $\tau := \tau_{K^m}$

$\tau' := \tau_{L^n}$

$$\tau \times \tau' \cong (\pi_1^* \tau) \oplus (\pi_2^* \tau')$$

$$\begin{tikzcd} K \times L \arrow[r, bend left, "\pi_1"] \arrow[dr, bend left, "\pi_2"'] & K \\ & L \end{tikzcd}$$

$$S_I[K \times L] = \langle S_I(\tau \times \tau'),\ [K \times L] \rangle$$

$$= \langle \underbrace{S_I(\pi_1^* \tau \oplus \pi_2^* \tau')}_{\text{Lemma}},\ [K] \times [L] \rangle$$

$$= \langle \sum_{I_1 I_2 = I} S_{I_1}(\pi_1^* \tau)\, S_{I_2}(\pi_2^* \tau'),\ [K^m] \times$$

$$= \sum_{\substack{I_1 I_2 = I \\ I_1 \vdash m,\ I_2 \vdash n}} \langle S_{I_1}(\tau), [K] \rangle \langle S_{I_2}(\tau'),\ [$$

$[L^n] >$

$[L] > \qquad \square$

# Week 13

Wednesday, December 7, 2022

Last time: **Algebra**: Symmetric function theory.

$$S = \mathbb{Z}[t_1, \dots, t_n]^{S_n}$$

2 bases for $S^k := \{\deg k \text{ parts}\} \subseteq S$

① $S \underset{\text{FTSP}}{=} \mathbb{Z}[\sigma_1, \dots, \sigma_n]$

take $\{\sigma_I : I \vdash k\}$

② take $\{m_I : I \vdash k\}$

where $m_I = \sum\limits_{\substack{\text{equivalent}\\ \text{monomials}}} t_1^{i_1} t_2^{i_2} \cdots t_r^{i_r}$ , $I = (i_1 \dots i_r)$

$m_I$ can be expressed using $\sigma_1, \dots, \sigma_n$.

$m_I = S_I(\sigma_1, \dots, \sigma_n)$

**Topology**: $H^*(BU(1)^{\times n}) \cong \mathbb{Z}[t_1, \dots t_n]$ $\quad |t_i| = 2$.

$H^*(BU(n)) \cong \mathbb{Z}[t_1 \dots t_n]^{S_n} = \mathbb{Z}[c_1, \dots, c_n]$

$c_i = \sigma_i$

Define $S_I(c(\omega)) := S_I(c_1(\omega), \dots, c_n(\omega))$ characteristic class of $\omega$.

**Cor.**: If $m, n \neq 0$, then $s_{m+n}[K^m \times L^n] = 0$

(because $(m+n) = (m+n)$ has no nontrivial juxtaposition).

**Ex.**: $(\mathbb{C}P^n)$.

$\tau = \tau_{\mathbb{C}P^n}$

$c(\tau) = (1+a)^{\underline{n+1}}$ $\quad a \in H^2(\mathbb{C}P^n)$

$\Rightarrow \forall k$, $c_k(\tau) = \sigma_k(\underbrace{a, \dots, a}_{n+1}) = \binom{n+1}{k} a^k$

$\quad \quad \quad \quad \; \; \| $

$\sigma_k[t_1, \dots, t_{n+1}]\big|_{t_i = a}$

$S_k(c_1, \dots, c_k) \overset{\text{def}}{=} m_k(a, \dots, a)$

Recall: $t_i$ formal

$(1+t_1)(1+t_2)\cdots$

$= 1 + \sigma_1 + \sigma_2 + \cdots$

$$\Big|_{t_i=a}$$

$$S_k(c_1, \dots, c_k) \overset{\text{def}}{=} m_k(\underbrace{a, \dots, a}_{n+1}) = m_k(t_1, \dots, t_{n+1})\Big|_{t_i=a}.$$

$$\uparrow \qquad \uparrow$$

$S_k(c_1(t_1, \dots, t_{n+1}), c_2 \dots)$

$\|$

$m_k(t_1, \dots, t_{n+1})$.

$$= \underbrace{a^k + a^k + \dots + a^k}_{(n+1)}$$

$$= (n+1)\, a^k \quad \leftarrow \text{char. class}$$

Take $n = k$ $\leftarrow$ char. number

$$S_n[\mathbb{C}P^n] = \langle S_n(T_{\mathbb{C}P^n}), [\mathbb{C}P^n] \rangle = (n+1) \overset{\in \mathbb{Z}}{\neq} 0$$

Cor: $\mathbb{C}P^n \neq K \times L$ unless $K$ or $L = \mathbb{C}P^n$

## Pontryagin classes

Apply similar ~~de~~ construction to Pontryagin classes of a real vector bundle $\xi$ over $B$.

$\forall I \vdash n$, $\quad S_I(p(\xi)) := S_I(p_1(\xi), \underset{\substack{\cap \\ H^{4n}(B;\mathbb{Z})}}{p_2(\xi)}, \dots, p_n(\xi))$

char. class $\rightarrow$ prop. $\quad S_I(p(\xi \oplus \xi')) = \sum_{JK=I} S_J(p(\xi))\, S_K(p(\xi')) + 2$ torsions.

char. number. $\rightarrow$ Cor: $\quad \overset{\mathbb{Z}}{\overset{\Downarrow}{S_I(p)[M \times N]}} = \sum_{JK=I} S_J(p)[M] \cdot \overset{\mathbb{Z}}{\overset{\Downarrow}{S_K(p)[N]}}$

Def: $S_I(p)[M] := \langle S_I(p(T_M)), [M] \rangle \in \mathbb{Z}$.

Rmk: (1) Cor is an honest equality because $\mathbb{Z}$ has no 2-torsion.

(2) $S_I(p)[M \times N] = 0$ unless $4 \mid \dim_{\mathbb{R}} M, \dim_{\mathbb{R}} N$

## Chern & Pontryagin numbers are linearly independent.

$\nearrow$ complex dim.

**Thm 1** (Thom). Let $K^1, K^2, \cdots, \overset{\text{complex dim.}}{K^n}$ are complex manifolds s.t.

$$S_k(c)[K^k] \neq 0 \quad \forall k=1,2,\cdots,n.$$

Then the matrix: $p(n)\times p(n)$ matrix.

$$\Phi_n = \left[\underbrace{c_{i_1}c_{i_2}\cdots c_{i_r}}\,[K^{j_1}\times\cdots\times K^{j_s}]\right]_{\substack{I\vdash n\\ J\vdash n}}$$

is nonsingular.

Rmk: ① Thm 1 $\Rightarrow$ $\{c_I : I\vdash n\}$ are independent as invariants of complex manifolds.

② recall $p(n) = \#\{I\vdash n\}$ "partition number".

Example: Take $K^k := \mathbb{CP}^k$. Recall: $S_k[\mathbb{CP}^k] = k+1 \neq 0$.

$\Updownarrow$ such family exists!

Example (Thm 1 for $n=2$)

partitions of 2 are $2=2$, $1+1=2$. $\quad p(2)=2$.

$K^k = \mathbb{CP}^k$. $k=1,2$.

$$\Phi_2 = \begin{bmatrix} c_1c_1[\mathbb{CP}^1\times\mathbb{CP}^1] = \text{⑧} & c_1c_1[\mathbb{CP}^2] = 9 \\ c_2[\mathbb{CP}^1\times\mathbb{CP}^1] = \text{④} & c_2[\mathbb{CP}^2] = 3 \end{bmatrix}$$

$\det\Phi_2 = -12 \neq 0$.

Similar results hold for Pontjagin classes.

**Thm 2** (Thom)

If $M^{4k}$, $(k=1,\cdots,n)$ are oriented manifolds with $S_k(p)[M^{4k}] \neq 0 \quad \forall k$

then the matrix

with $s_k(p)[M^{4k}] \neq 0$ ∀k

then the matrix

$$\left[ p_{i_1} \cdots p_{i_r} [ M^{4j_1} \times \cdots \times M^{4j_s} ] \right]_{\substack{I \vdash n \\ J \vdash n}}$$

is nonsingular.

Ex. $M^{4k} := \mathbb{C}P^{2k}$ Before $p(\mathbb{C}P^{2k}) = (1+a^2)^{2k+1}$

(You)

$$\Rightarrow \quad s_k(p)[\mathbb{C}P^{2k}] = 2k+1 \neq 0 \quad \forall k.$$

pf of Thm 1. Note $\{ \overset{\sigma_I}{\overset{\shortparallel}{c_I}} \mid I \vdash n \}$ and $\{ \overset{m_I}{\overset{\shortparallel}{s_I(c)}} \mid I \vdash n \}$

are both bases for $H^{2n}(BU(n);\mathbb{Z})$

After a change of bases, it suffices to show that

$$\Big[ \underbrace{s_I(K^{j_1} \times \cdots \times K^{j_s})}_{(*)} \Big]_{\substack{I \vdash n \\ J \vdash n}} \text{ is nonsingular.}$$

$$\underset{\substack{\uparrow \\ \text{product} \\ \text{formula}}}{(*) =} \sum_{I_1 \cdots I_s = I} s_{I_1}[K^{j_1}] \cdots s_{I_s}[K^{j_s}]$$

$= 0$ unless $I_k \vdash j_k$ ∀$k = 1, \dots, s$

or $I$ is a refinement of $J$.

Hence, the matrix

$$\left[ s_I[K^J] \right]_{\substack{I \vdash n \\ J \vdash n}} \text{ is upper triangular}$$

if we order $I, J$ by refinement.

Hence,

$$\det[\ \downarrow\ ] = \prod s_I[K^I].$$

Hence,

$$\det[\downarrow] = \prod_{\substack{I=J \\ I \vdash n}} \underbrace{s_I[k^I]}_{\substack{\neq \\ 0}} \neq 0 \ |$$

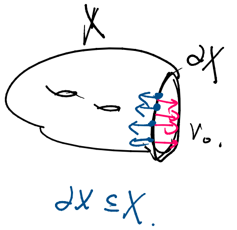
each $s_I[k^I] = s_{i_1 \cdots i_r}[k^{i_1} \times \cdots \times k^{i_r}]$

$$\overset{\text{product formula}}{=} \ \underset{\substack{\neq \\ 0}}{s_{i_1}[k^{i_1}]} \ \cdots \ \underset{\substack{\neq \\ 0.}}{s_{i_r}[k^{i_r}]} \neq 0$$

$\square$.

## Oriented cobordism ring $\Omega_*$ ( Thom )

Reference: M-S § 17.



Setup: $(X^{n+1}, \partial X)$ smooth manifold with boundary

orientation on $TX \rightsquigarrow$ orientation on $T(\partial X)$

$\forall x \in \partial X$, a basis $v_1, \dots, v_n \in T_x(\partial X)$ is positively oriented

if $\exists\ v_0 \in T_x X \setminus T_x(\partial X)$ outward

s.t. $v_0, v_1, \dots v_n$ is positively oriented for $T_x X$.

Notation: $M$ = smooth closed oriented manifold

$-M :=$ $M$ with opposite orientation.

$M + M' := M \amalg M'$.

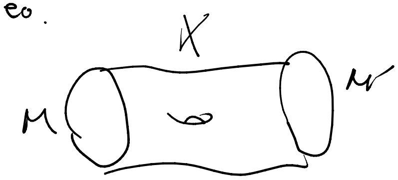
Def: Two smooth closed oriented $n$-manifolds $M$ and $M'$ are oriented cobordant

if $\exists$ smooth (connected) compact oriented manifold with boundary $X$

s.t. $\partial X \cong M + (-M')$

s.t. $\partial X \overset{\uparrow}{\cong} M + (-M')$

orientation-preserving diffeo.



Rmk: • Being oriented cobordant is an equivalence relation.

• Equivalence class is called "cobordism class".

$$\Omega_n := \{ [M^n] \mid M^n \text{ smooth, closed, oriented} \}.$$

forms an abelian group under +.

$$\Omega_* := \bigoplus_{n \geq 0}^{\infty} \Omega_n \text{ forms a graded ring}$$

under $[M] \times [N] := [M \times N]$.

note: $[M^m \times N^n] = (-1)^{mn} [N^n \times M^m]$

$\Omega_*$ is graded commutative.

Recall: (HW): If $M^{4k} = \partial V^{4k+1}$

then $p_I[M] = 0 \quad \forall I \vdash k$.

Cor: For any $I \vdash k$, we have a group homomorphism

$$\begin{aligned} p_I : \Omega_{4k} &\longrightarrow \mathbb{Z} \\ [M^{4k}] &\longmapsto p_I[M^{4k}] \end{aligned}$$

pf: you.

Cor. The set

$$\{ \mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_r} \mid I \vdash k \} \subseteq \Omega_{4k}.$$

are $\mathbb{Z}$-independent in $\Omega_{4k}$.

Hence, $\operatorname{rank}(\Omega_{4k}) \geq p(k)$

[illegible] in $\Omega_{4k}$.

Hence, $\text{rank}(\Omega_{4k}) \geq p(k)$.

Pf. The matrix $[p_I(\mathbb{C}p^J)]_{I,J}$. is nonsingular $\square$.
(Thm 2)

Next, we will prove $\text{rank}(\Omega_{4k}) = p(k)$.

Examples: oriented cobordism.

$\Omega_0 \cong \mathbb{Z} = \mathbb{Z}\{[*]\}$

$\Omega_1 = 0$  $S^1 = \partial D^2$

$\Omega_2 = 0$ 

$\Omega_3 = 0$ (Rohlin)

$\to \Omega_4 \cong \mathbb{Z} = \mathbb{Z}\{[\mathbb{C}P^2]\}$ (pf later.)

$\Omega_5 \cong \mathbb{Z}/2\mathbb{Z} = \langle Y \rangle$

$\Omega_6 = 0$

$\Omega_7 = 0$

$\to \Omega_8 = \mathbb{Z} \oplus \mathbb{Z}$. $rk = 2 = p(2)$

$\Omega_9 = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

$\Omega_{10} = \mathbb{Z}/2$

$\vdots$

$\Omega_* = ?$

Week 14

Wednesday, December 14, 2022

**Thm**$^{*}$ (Thom)

$$\Omega_n = \begin{cases} \text{finite} & \text{if } n \not\equiv 0 \mod 4 \\ \mathbb{Z}^{p(k)} \oplus \text{finite} & \text{if } n = 4k. \end{cases} \qquad \left( \begin{array}{l} p(k) = \#\text{ of partitions of } k \\ \quad = \#\{I \vdash k\} \end{array} \right)$$

**Cor 1.** $\Omega_* \otimes \mathbb{Q} \cong \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \cdots]$

**Cor 2.** For $M^n$ smooth closed oriented,

$M^n + \cdots + M^n = \partial V^{n+1}$ $\quad$ ( $\underset{0}{\underset{\shortparallel}{[M^n]}} \in \Omega_n \otimes \mathbb{Q}$ ~~is torsion~~ ).

iff every Pontryagin number of $M$ is zero.

**Thm.** (Wall). $M^n = \partial V^{n+1}$

iff every Pontryagin number and Stiefel-Whitney number are zero.

**pf idea:** Pontryagin's work on the framed cobordism groups.

**Def.** A framed $k$-manifold in $\mathbb{R}^{n+k}$ is a closed

$$M^k \hookrightarrow \mathbb{R}^{n+k}$$

with a trivialization of normal bundle.

$M \sim M'$ iff $\exists$ framed $(k+1)$ manifold $V^{k+1} \hookrightarrow \mathbb{R}^{n+k} \times [0,1]$

framed cobordant

s.t. $\partial V = M \sqcup M'$

and the framing on $V$ restricts to those on $M, M'$.

$$\Omega_k^{fr}(\mathbb{R}^{n+k}) = \{ [M^k] \mid M^k \subseteq \mathbb{R}^{n+k} \text{ framed} \}.$$

**Thm** (Pontryagin) $\forall n \geq 1$, $\forall k \geq 0$,

$S^n = (\mathbb{R}^n)^+$
$= \mathbb{R}^n \cup \{\infty\}$.

$$\Omega_k^{fr}(\mathbb{R}^{n+k}) \xleftarrow{\cong} \pi_{n+k}(S^n)$$

$$[f^{-1}(p)] \longleftarrow\!\!\shortmid \quad f : S^{n+k} \longrightarrow \overset{p}{\underset{}{S^n}} \quad \text{smooth}$$

$$\infty \longmapsto \infty$$

$k$-manifold in $\mathbb{R}^{n+k}$ $\qquad$ $p \in \mathbb{R}^n$, $p \neq \infty$, $\quad f \pitchfork p$.

Framing on $T_p(\mathbb{R}^n)$ ⇝ Framing on normal bundle of $f^{-1}(p) \subseteq \mathbb{R}^{n+k}$.

Difficulty: compute $\pi_*(S^n)$

Thom's insight: similar result holds for $\Omega_k$.

$$\Omega_k \cong \pi_{n+k}(T_\cdot)$$

↳ easier to compute

$$\Omega_k \overset{\cdot\cdot}{=} \pi_{n+k}(T_\cdot)$$

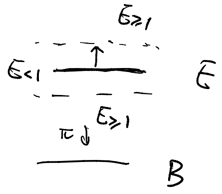
↳ much easier to compute

## pf sketch of Thom's cobordism thm

### (I) Thom space of a vector bundle.

→ $\xi$ = real vector bundle with an Euclidean metric

**Def**: The Thom space of $\xi$ is



$$\boxed{T = T(\xi) := E/E_{\geq 1}}$$

$$\begin{tikzcd} E \arrow[r] & T = E/E_{\geq 1} \\ E_{\geq 1} \arrow[u, hook] \arrow[r] & t_0 \arrow[u, hook] \end{tikzcd}$$

where $E_{\geq 1} = \{ e \in E \mid |e| \geq 1 \}$.

So $T(\xi) \setminus t_0 = E \setminus E_{\geq 1} = E_{<1}$.

**Rmk**: If $B$ is compact, then $T(\xi) \cong E \cup \{\infty\}$.

↖ 1-point compactification of $E$.

### Sketch of Thom's computation of $\Omega_n$

→ **Step 1**. For $\xi$ an $k$-plane bundle, define a map

(DT).

$$\begin{aligned} \pi_{n+k}(T, t_0) &\longrightarrow \Omega_n \\ (g : S^{n+k} \to T) &\longmapsto [g^{-1}(B)] \end{aligned}$$

$g \pitchfork B \subseteq T$. dim=n

→ **Step 2**. when $\xi = \tilde{\gamma}^k$ over $\tilde{G}_k = BSO(k)$, → oriented Grassmanian.

(DT) the map above is an isomorphism when $k$ is large.

MSO(k) ⊗ℚ        BSO(k) ⊗ℚ

√ → **Step 3**. Show $\pi_{n+k}(T(\tilde{\gamma}^k)) \cong H_n(\tilde{G}_k; \mathbb{Z})$ modulo torsions.

(AT). Then apply our knowledge of ⤴

We will first do step 3. and then step 1, 2.

---

**Lemma 1**. If $B$ is a CW complex          $k = \dim \xi$

then $T(\xi)$ is a $(k-1)$-connected CW complex,

Moreover, each $n$-cell of $B$ gives ~~an~~ an $(n+k)$-cell of $T(\xi)$.

Rmk If $B$ is finite, then so is $T$.

an $(n+k)$-cell of $T(\xi)$.

Rmk. If $B$ is finite, then so is $T$.

pf sketch. For each $n$-cell $e_\alpha : D^n \longrightarrow B$

we have:

$$\begin{tikzcd} D^n \times D^k \arrow[r, "\tilde{e}_\alpha"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_{\le 1} \arrow[r, two heads] \arrow[d] & \frac{E_{\le 1}}{E_{=1}} = T = \frac{E}{E_{\ge 1}}. \\ D^n \arrow[r, "e_\alpha"] & B & \end{tikzcd}$$

$\tilde{e}_\alpha$ gives an $(n+k)$-cell for $T$. □.

Lemma 2. If $\xi$ is an oriented $k$-plane bundle over $B$,

then $H_{k+i}(T(\xi), t_0) \cong H_i(B)$ $\forall i$.

pf sketch. The zero section $B \hookrightarrow E_{<1} = T \setminus \{t_0\}$

$T_0 := T \setminus B$

note: $T_0 \simeq t_0$.

so $H_\bullet(T, t_0) \cong H_\bullet(T, T_0)$

excision: $H_\bullet(T, t_0) \cong H_\bullet(E, E_0)$

Thom isomorphism Thm: $H_{k+i}(E, E_0) \cong H_k(B)$ □.

Recall: $\mathcal{C} = \{$ finite abelian groups $\}$.

A homomorphism $h : A \to B$ of ab. gps. is a $\mathcal{C}$-isomorphism if $\ker(h)$, $\operatorname{coker}(h) \in \mathcal{C}$.

Thm 3 (Serre).

Suppose $X$ is a finite CW complex, and $(k-1)$-connected, $k \geq 2$,

then the Hurewicz map $\pi_r(X) \longrightarrow H_r(X; \mathbb{Z})$ is an $\mathcal{C}$-iso. for $r < 2k-1$.

pf idea:

Step 1: Thm true for $X = S^n$, $n \geq k$.

pf idea:

Step1: Thm true for $X = S^n$, $n \geqslant k$.

since $\pi_r(S^n)$ is finite $\forall\ r < 2n-1$, $r \neq n$. (Serre).

Step2: Thm true for wedge of spheres. (cellular approx.).

Step3: Consider general $X$.

key: $X$ finite CW, $\pi_1(X) = 0 \Rightarrow \pi_r(X)$ finitely generated $\forall r$.

(Serre's generalized Hurewicz).

pick a basis for the torsion-free part of $\pi_r(X)$, $r < 2k$.

$$\{ f_{r_i} : S^{r_i} \to X \}_{i=1}^{N}$$

$$\rightsquigarrow f : \bigvee_{i=1}^{N} S^{r_i} \longrightarrow X .$$

step2: Thm true ↑  ↑ $\mathcal{C}$-iso on $\pi_r$ Generalized Hurewicz. ⇒ Thm true for $X$.

Cor 4. For $\xi$ an oriented $k$-plane bundle over a finite CW $B$

$T := T(\xi)$  $\mathcal{C} = \{\text{finite ab. grps}\}$.

then we have a $\mathcal{C}$-iso.

$$\pi_{n+k}(T) \longrightarrow H_n(B, \mathbb{Z}) \quad \forall\ n < k-1,$$

Pf: $B$ finite CW $\xRightarrow{\text{Lemma 1}}$ $T$ finite CW, $(k-1)$-connected.

Thm 3. $\Rightarrow$ $\mathcal{C}$-iso.

$$\begin{tikzcd} \pi_{n+k}(T) \arrow[r] & H_{n+k}(T;\mathbb{Z}) \arrow[d, phantom, "\cong"', "\text{Lemma 2}"] & \forall\ n < k-1. \\ & H_n(B;\mathbb{Z}) & \end{tikzcd}$$

□

---

Thom space v.s. cobordism. (Step 1, 2).

Let $\zeta$ = smooth oriented $k$-plane bundle over a smooth manifold $B$.

Let $\xi$ = smooth oriented k-plane bundle over a smooth manifold $B$.
(E has a smooth str.).

$B \hookrightarrow E$ via the zero section.

$T = T(\xi)$.

For any $[f] \in \pi_m(T, t_0)$.

pick a "smooth" representative $f: S^m \to T$.

( note: $T = E/E_{\geq 1}$ is not a smooth manifold

but $T - t_0 \cong E_{<1}$ is — - —

f "smooth" means

$f^{-1}(T \setminus t_0) \xrightarrow{f} T \setminus t_0$ is smooth )

Moreover, pick such f that is transverse to the zero section $B \hookrightarrow T \setminus t_0$.

i.e. every $b \in B$ is a regular value of f

i.e. $\forall x \in f^{-1}(b)$, $T_x(S^m) \xrightarrow{f_*} T_b(T \setminus t_0)$ is onto.

transversality

$f \pitchfork B \Rightarrow f^{-1}(B)$ is a smooth submanifold of $S^m$.

↘ dim = m-k.

$\xi$ oriented $\Rightarrow$ $f^{-1}(B)$ oriented.

Thm. The construction above gives a homomorphism

$$\pi_m(T, t_0) \longrightarrow \Omega_{m-k}.$$
$$[f] \longmapsto [f^{-1}(B)]$$

Pf: (well-defined)

if $f, g: S^m \to T$ are homotopic

and smooth and $\pitchfork B$

Let $h: S^m \times [0,1] \longrightarrow T$ be a homotopy.

s.t. $h_0 = f$. $h_1 = g$. $\pitchfork B$.

s.t. $h_0 = f$. $h_1 = g$. $\pitchfork B$.

We can homotope $h$ relative to $\partial$ s.t.

1. $h$ is smooth on $T - t_0$
2. $h \pitchfork B$.

Then $h^{-1}(B)$ is a submanifold with $\partial$ in $S^m \times [0,1]$

$\rightsquigarrow \partial(h^{-1}(B)) = h_0^{-1}(B) + [-h_1^{-1}(B)]$ . . .

→ **Thm** (Thom) For $k > n+1$

$$\to \pi_{n+k}(T(\tilde{\gamma}^k), t_0) \xrightarrow{\cong} \Omega_n \quad \text{oriented cobordism group.}$$

$$\to \pi_{n+k}(T(\gamma^k), t_0) \xrightarrow{\cong} \mathcal{N}_n \quad \text{cobordism group.}$$

We will sketch proof of a weaker result:

**Prop.** If $k \geq n$, $p \geq n$, then

$$\pi_{n+k}(T(\tilde{\gamma}^k_p)) \longrightarrow \Omega_n \text{ is onto.}$$

Recall:

$$\begin{tikzcd} \gamma^k_p \arrow[d] \\ G_k(\mathbb{R}^{p+k}) \end{tikzcd}$$

Pf: Pick $[M^n] \in \Omega_n$ $M^n$ smooth oriented $n$-manifold

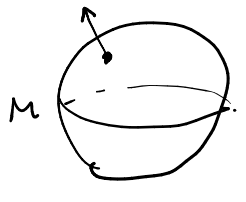
Whitney's embedding thm $\Rightarrow$ $M^n \hookrightarrow \mathbb{R}^{2n} \subseteq \mathbb{R}^{n+k}$

Consider $M^n \subseteq \mathbb{R}^{n+k}$

Gauss map:

$$\begin{tikzcd} (x, v) \arrow[rr, mapsto] & & ((T_xM)^\perp, v) \\ E(\nu^k) \arrow[u, phantom, "\in" description] \arrow[rr, "g"] \arrow[d] & & E(\tilde{\gamma}^k_n) \arrow[d] \\ M \arrow[rr] & & \tilde{G}_k(\mathbb{R}^{n+k}) = B \end{tikzcd}$$

linear and injective on fibers

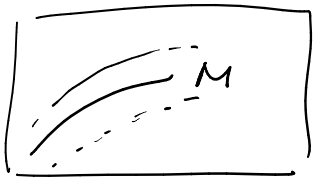


fibers

$$\begin{tikzcd} M \arrow[r] & G_k(\mathbb{R}^{n+k}) = B \\ x \arrow[r, mapsto] & \nu^k(x) = (T_xM)^\perp \end{tikzcd}$$

note: $g^{-1}(B) = M$.

↳ zero section of $\tilde{\gamma}^k_n$



Let $U$ be a tubular neighborhood of $M$ in $\mathbb{R}^{n+k}$.

i.e. $U \cong E(\nu^k)$

$$\begin{tikzcd} U \cong E(\nu^k) \arrow[r, "g"] \arrow[rr, bend right, "f"'] & E(\tilde{\gamma}^k_n) \subseteq & E(\tilde{\gamma}^k_p) \end{tikzcd}$$ if $p \geq n$.

$M \subseteq \mathbb{R}^{n+k} \subseteq S^{n+k}$.

$\subseteq U \subseteq$

Extend $f$ to $S^{n+k}$ by mapping $S^{n+k} \setminus U$ to to

$$\hat{f} : S^{n+k} \longrightarrow T(\tilde{\gamma}^k_p)$$

$$\hat{f}^{-1}(B) = f^{-1}(B) = M$$

So
$$\begin{tikzcd} \pi_{n+k}(T) \arrow[r] & \Omega_n \\ \hat{f} \arrow[r, mapsto] & {[M]} \end{tikzcd}$$

□

**Summary:** We have:

$\boxed{\operatorname{rank} \Omega_{4k} \leq p(k)).}$

$\boxed{\text{Before: } \operatorname{rank} \Omega_{4k} \geq p(k)).}$

$$\pi_{n+k}(T(\tilde{\gamma}^k_p)) \twoheadrightarrow \Omega_n \qquad (p, k \geq n)$$

≅ $\mathcal{C}$-iso.

$$H_n(\tilde{G}_k(\mathbb{R}^{k+p}); \underset{\sim}{\mathbb{Z}})$$

note: $\tilde{G}_k(\mathbb{R}^{n+k})$ is a finite CW complex. ✓

$$\hookrightarrow = \begin{cases} \underset{\sim}{\text{finite}} & n \not\equiv 0 \bmod 4 \\ \mathbb{Z}^{p(\frac{n}{4})} \oplus & n \equiv 0 \bmod \end{cases}$$

$$\Big| \ \mathbb{Z}^{p(\frac{n}{4})} \oplus \text{finite} \quad n \equiv 0 \bmod 4.$$

Recall: $H^*(\widetilde{G}_k ; \Lambda) = \Lambda[p_1, p_2, \cdots]/\sim$

$\hookrightarrow \frac{1}{2} \in \Lambda.$ (e.g. $\Lambda = \mathbb{Q}$).

$$\Rightarrow \quad \Omega_n = \begin{cases} \text{finite} & n \not\equiv 0 \bmod 4. \\ \mathbb{Z}^{p(\frac{n}{4})} \oplus \text{finite} & n \equiv 0 \bmod 4 \end{cases}$$

$$\Omega_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \cdots]$$

## Hirzebruch's Signature Theorem

Consider $M^{4k}$ closed, oriented.

Poincaré pairing: $H^{2k} := H^{2k}(M^{4k}; \mathbb{Q})$.

$$H^{2k} \times H^{2k} \xrightarrow{P} \mathbb{Q}$$

$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta, [M^{4k}] \rangle$$

$$P(\alpha, \beta) = \underbrace{(-1)}_{+}{}^{|\alpha||\beta|} P(\beta, \alpha)$$

$\Rightarrow P$ is a symmetric bilinear form (non-degenerate)

signature of $P$ is:

$\sigma(P) :=$ # of positive eigenvalues of $P$

− # .. negative .. .. .. .. ..

Def: The signature of $M^{4k}$ is $\sigma(M) := \sigma(P_M)$

If $4 \nmid \dim_{\mathbb{R}} M$, we say $\sigma(M) = 0$.

<u>Lemma</u>. (Thom).

(1) $\sigma(M + M') = \sigma(M) + \sigma(M')$

(2) $\sigma(M \times M') = \sigma(M)\,\sigma(M')$

(3) if $M = \partial V$, $V$ compact oriented
then $\sigma(M) = 0$

pf: (1).

(2) Kunneth.

(3): Poincaré duality for $(V, \partial V)$ exercise.

<u>Cor</u>. The map $M \longmapsto \sigma(M)$ gives a ring homomorphism

$$\Omega_\bullet \longrightarrow \mathbb{Z},$$

<u>Recall</u>: By Thom's cobordism thm.

$$\Omega_\bullet \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \dots]$$

$$\sigma \in \left(\Omega_\bullet \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\right)^\vee = \mathbb{Q}[p_1, p_2, \dots]$$

Hence, $\sigma$ can be expressed in terms of Pontryagin classes.

Hirzebruch's thm $\Rightarrow$ a formula for $\sigma$ using $p_i$'s.

<u>Multiplicative sequences</u> (algebra).

$\Lambda :=$ commutative ring with $1$ (e.g. $\Lambda = \mathbb{Q}$).

$A^* := \bigoplus_{n=0}^{\infty} A^n$ graded $\Lambda$-algebra. (e.g. $A^n = H^{4n}(M;\Lambda)$)

$$A^{\Pi} := \prod_{n=0}^{\infty} A^n = \left\{ \underbrace{\underset{\substack{\downarrow\\ \shortparallel \\ 1}}{a_0} + a_1 + a_2 + \cdots}_{a} \;\middle|\; a_n \in A_n \ \forall n \right\}$$

Consider polynomials

$$\prod_{n=0} \quad \Big| \; \underbrace{\qquad + a_2 + \cdots}_{a} \; \Big| \; a_n \in A_n \;\; \forall n \Big\}$$

$= 1$

Consider polynomials

$$K_1(x_1), \quad K_2(x_1, x_2), \quad K_3(x_1, x_2, x_3), \; \cdots\cdots$$

with coefficients in $\Lambda$ s.t.

if $\deg x_i = i$, then $K_n$ is homogeneous of deg $n$.

Given $a \in A^{\Pi}$,

define $K(a) := 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots \quad \in A^{\Pi}$

**Def.** $K_n$ forms a multiplicative sequence of polynomials

if $K(ab) = K(a)\,K(b)$

holds for all $a, b \in A^{\Pi}$ with leading coefficient 1

for all graded $\Lambda$-algebra $A^*$.

**Ex 1:** $\lambda \in \Lambda$, Define $K_n = \lambda^n x_n$ belong to $f(t) = 1+\lambda t$.

$$K(1 + a_1 + a_2 \cdots) = 1 + \lambda a_1 + \lambda^2 a_2 + \cdots\cdots$$

is a multiplicative sequence.

eg. If $\omega$ is a complex vector bundle,

then $c(\overline{\omega}) = 1 - c_1(\omega) + c_2(\omega) - \cdots$ $\quad c(\omega) \in H^{\Pi}$.

$= K(c(\omega))$ where $\lambda = -1$.

**Ex 2:** $K(a) = a^{-1}$. is a multiplicative sequence. belong to $f(t) = 1 - t + t^2 - t^3 + \cdots$

note: $a = 1 + a_1 + a_2 \cdots$ in $A^{\Pi}$ has an inverse $a^{-1} \in A^{\Pi}$.

$K_1(x_1) = -x_1$

$K_2(x_1, x_2) = x_1^2 - x_2$.

$K_2(x_1, x_2) = x_1^2 - x_2$.

$\vdots$

$K_n(x_1, \dots, x_n) =$ inductive formula.

e.g. If $\xi \oplus \eta = \varepsilon^N$, then $c(\xi)\, c(\eta) = 1$

$c(\eta) = c(\xi)^{-1} = K(c(\xi))$.

Ex 3: $K_{2n+1} := 0$ belong to $f(t) = 1 + t^2$. $(1 - t^2 ?)$.

$K_{2n}(x_1, \dots, x_{2n}) := x_n^2 \underset{\uparrow}{-} 2x_{n-1}x_{n+1} \underset{\uparrow}{+} \cdots \pm 2x_1x_{2n-1} \mp 2x_{2n}$

$K$ is a multiplicative seq.

e.g. If $\omega$ is a complex v.b.

then $p(\omega_{\mathbb{R}}) = K(c(\omega))$

Classification of multiplicative sequences. $K$.

Let $A^* := \Lambda[t]$ with $\deg(t) = 1$.

$A^\Pi = \{ f(t) = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots \} = \Lambda[[t]]$

e.g. $1 + t \in A^\Pi$.

Lemma (Hirzebruch).

$\forall f(t) \in \Lambda[[t]]$,

$\exists$ unique multiplicative sequence $\{\underset{\uparrow}{K_n}\}$ s.t.

$K(1+t) = f(t)$.

Equivalently, $\forall n$, each $K_n(x_1, \dots, x_n)$ satisfies that the coefficient of $x_1^n$ is $\lambda_n \in \Lambda$

the coefficient of $x_1^n$ is $\lambda_n \in \Lambda$

↪ $f(t) = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots$

---

In this case, we say $\{K_n\}$ is the multiplicative seq. belonging to the power series $f(t)$

pf of lemma. (uniqueness).

Take $A^* := \Lambda[t_1, \dots, t_n]$ $\deg t_i = 1$. $\forall i$.

$$\sigma := (1+t_1) \cdots (1+t_n)$$
$$= 1 + \sigma_1 + \sigma_2 + \cdots + \sigma_n .$$

Then
$$K(\sigma) = \underbrace{K(1+t_1)}_{\overset{\shortparallel}{f(t_1)}} \cdots \underbrace{K(1+t_n)}_{\overset{\shortparallel}{f(t_n)}}$$
$$\overset{(*)}{=} f(t_1) \cdots f(t_n)$$

Hence, $K_n(\sigma_1, \dots, \sigma_n)$ on the LHS is determined by $f(t)$ on the RHS. ⇒ $K$ unique.

(existence). Use $(*)$ to define $K$.

$$K(\sigma) = \sum_n K_n(\sigma_1, \dots, \sigma_n) := \prod_i f(t_i)$$

↪ n-th homogeneous part.

Let
$$K_n(\sigma_1, \dots, \sigma_n) = \sum_{I \vdash n} \lambda_I \, s_I(\sigma_1, \dots \sigma_n)$$

where $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_r}$

Recall $f(t) = \lambda_0 + \lambda_1 t + \lambda_2 t^2 + \cdots$

$$s_I(\sigma_1, \dots, \sigma_n) = m_I(t_1 \cdots t_n)$$
$$= \sum \cdot^{i_1} \quad \cdot^{i}$$

$$I \ldots n) = m_I(t_1 \dots t_n)$$

$$= \sum_{\text{all equivalent.}} t_1^{i_1} \dots t_r^{i_r}$$

<u>check</u>: $K(ab) = K(a)K(b)$. $\leftarrow$

using the product formula (Thom):

$$S_I(ab) = \sum_{I = JK} S_J(a) S_K(b), \quad \forall a, b \in A^{\Pi}$$

$\square$.

# Week 15

Wednesday, December 21, 2022

## Signature Theorem (topology).

Take $\Lambda := \mathbb{Q}$

$K :=$ a multiplicative sequence of poly. over $\mathbb{Q}$.

Let $M^m :=$ smooth closed oriented $m$-manifold.

Def. The K-genus of $M^{4n}$ is

$$K_n[M^{4n}] := \langle \underset{H^{4n}(M^{4n};\mathbb{Q})}{\underset{\in}{K_n(p_1, \cdots, p_n)}} , \underset{H_{4n}}{\underset{\in}{[M]}} \rangle \in \mathbb{Q}.$$

If $m \not\equiv 0 \bmod 4$, then we set $K[M^m] = 0$.

Rmk. $K_n[M]$ only depends on $[M] \in \Omega_\bullet$.

Lemma. For any multiplicative sequence $\{K_n\}$ over $\mathbb{Q}$, the map

$$\Omega_* \xrightarrow{K} \mathbb{Q}$$
$$[M] \longmapsto K[M]$$

is a ring homomorphism.

Pf (You). $K$ is multiplicative.

Recall: signature $\Omega_* \xrightarrow{\sigma} \mathbb{Q}$ is a ring homo.

Thm (Hirzebruch's Signature Thm).

For any $M^{4k}$ smooth closed, oriented,

$$\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\sigma(M^{4k}) = L[M^{4k}]} \qquad \text{“}L\text{-genus”}$$

where $\{L_k\}$ is the multiplicative sequence of polynomials over $\mathbb{Q}$ belonging to the power series:

$$\rightarrow f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{k-1}\frac{2^{2k}}{(2k)!}B_k t^k$$

$B_k =$ k-th Bernoulli number.

Cor. • $\sigma(M^4) = L_1[M^4] = \frac{1}{3}p_1[M^4] \in \mathbb{Z}$.

$$\Rightarrow p_1[M^4] \in 3\mathbb{Z}.$$

• $\sigma(M^8) = L_2[M^8] = \frac{1}{45}(7p_2 - p_1^2)[M^8] \in \mathbb{Z}$

$$\Rightarrow 7p_2[M^8] - p_1^2[M^8] \in 45\mathbb{Z}.$$ (we will use later).

• $\sigma(M^{12}) = \frac{1}{945}(62p_3 - 12p_2p_1 + 2p_1^3)[M^{12}]$.

$$\Rightarrow \Big(\text{if } \underset{\underset{p_1=0.}{\Downarrow}}{H^4(M^{12})=0}, \text{ then } 62 \mid \sigma(M^{12}).\Big)$$

⋮

"integrality thms".

pf of Thm:

$$\Omega_\bullet \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$$
$$[M] \longmapsto \sigma[M]$$
$$[M] \longmapsto L[M].$$

both are ring maps.

$$[M] \longmapsto \sigma[M]$$
$$[M] \longmapsto L[M].$$

Thom's cobordism thm $\Rightarrow$ $\Omega_{\bullet} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \cdots]$.

It suffices to check that $\underset{\overset{\|}{+1}}{\sigma[\mathbb{C}P^{2k}]} = L_k[\mathbb{C}P^{2k}]$.

Goal: $L_k[\mathbb{C}P^{2k}] = +1$. $\forall k$.

recall. $p := p(T_{\mathbb{C}P^{2k}}) = (1+a^2)^{2k+1}$, $a \in H^2$.

$\{L_k\}$ belongs to $f(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$.

$\Rightarrow L(1+a^2) = f(a^2)$

$p = 1+p_1+p_2+\cdots \in H^{\Pi}$

$$L(p) = L((1+a^2)^{2k+1}) \quad \leftarrow L \text{ is multiplicative.}$$
$$= L(1+a^2)^{2k+1}.$$
$$= f(a^2)^{2k+1}$$
$$= \left(\frac{a}{\tanh a}\right)^{2k+1}.$$

By def. the L-genus,

$$L_k[\mathbb{C}P^{2k}] = \langle L_k(p), [\mathbb{C}P^{2k}] \rangle$$
$$= \underbrace{\text{coefficient}} \text{ of } a^{2k} \text{ in the power series } \left(\frac{a}{\tanh a}\right)^{2k+1}.$$

Goal: $= +1$. $\qquad \langle a^{2k}, [\mathbb{C}P^{2k}] \rangle = +1$.

To compute the coefficient, replace $a$ by a complex variable $z$.

$$z^{2k} \text{ coefficient in } \left(\frac{z}{\tanh z}\right)^{2k+1} = \frac{1}{2\pi i} \oint \left(\frac{z}{\tanh z}\right)^{2k+1} \frac{dz}{z^{2k+1}}.$$
$$= \frac{1}{2\pi i} \oint \frac{dz}{(\tanh z)^{2k+1}}$$

$$\left( \text{substitute: } \begin{array}{l} u := \tanh z \\ dz := \dfrac{du}{1-u^2} = (1+u^2+u^4+\cdots)du \end{array} \right)$$

$$= \frac{1}{2\pi i} \oint \frac{(1+u^2+u^4+\cdots)}{u^{\underbrace{2k+1}_{\text{odd}}}} du.$$
$$= +\underline{1}. \qquad \square.$$

## Milnor's Exotic $S^7$

Let $S^n := \{ x \in \mathbb{R}^{n+1} \mid |x| = 1 \}$ smooth. "standard $S^n$"

A homotopy n-sphere is a topological manifold $M^n$ that is homotopy equivalent to $S^n$.

A topological n-sphere - - - - - - - - - - - - - - - homeomorphic to $S^n$

A homotopy $n$-sphere is a topological manifold $M^n$ that is homotopy equivalent to $S$.
A topological $n$-sphere - - - - - - - - - - - - - - - - - homeomorphic to $S^n$
A differential $n$-sphere .. smooth - - - - - - - - - - diffeomorphic to $S^n$.

Generalized Poincaré Conjecture:

Every homotopy $n$-sphere in $\mathcal{C}$ is $\mathcal{C}$-isomorphic to the standard $S^n$.

$\mathcal{C}$ = TOP, PL (piecewise-linear), DIFF.

- TOP. true $\forall n$.
- PL. true $\forall n \neq 4$, open $n=4$ (equivalent to DIFF).
- DIFF. true $n = 1, 2, 3, 5, 6$.
  open $n=4$.
  false in general. (today: $n=7$).

Thm (Milnor). There exists a smooth 7-manifold that is homeomorphic, but not diffeomorphic, to standard $S^7$.

Pf sketch: (I) For each $(h,j) \in \mathbb{Z} \times \mathbb{Z}$, construct bundles:

$$\begin{tikzcd} S^3 \arrow[r] & M_{h,j} \arrow[d] \\ & S^4 \end{tikzcd}$$

$\to$ (II) $M_{h,j}$ is homeo. to $S^7$ if $h+j = \pm 1$ (Morse theory).

$\to$ (III) $M_{h,j}$ is NOT diffeo. to $S^7$ if $(h-j)^2 \not\equiv 1 \pmod 7$.
(Signature Thm).

Take e.g. $(h,j) = (3,-2)$

Convention: • All manifolds are assumed to be smooth, compact, oriented.
• Notation: $\approx$ means "diffeomorphic to".

## (I) $S^3$-bundles over $S^4$

$$G \to \underset{\ast}{\underset{\simeq}{EG}} \to BG$$

$$\left\{\begin{array}{l}\text{oriented rank-4} \\ \text{real vector bundles} \\ \mathbb{R}^4 \to E \to S^4 \\ \cup\!| \\ S^3 \end{array}\right\} \leftrightarrow [S^4, BSO(4)] \cong \pi_4(BSO(4)) \cong \underset{\overset{\rotatebox{90}{$\in$}}{f_{h,j}}}{\pi_3(SO(4))}.$$

$$\xi_{hj} \longleftarrow\!\shortmid f_{h,j}$$

Fact: This map is an iso.

Consider a map
$$\mathbb{Z} \times \mathbb{Z} \xrightarrow{\cong} \pi_3 SO(4)$$
$$(h,j) \longmapsto \left(f_{hj} : S^3 \to SO(4)\right)$$

where $\overset{\overset{SO(4)}{\rotatebox{90}{$\in$}}}{f_{hj}(u)} \cdot v := \underbrace{u^h v u^j}$ $\longrightarrow$ multiplication in $\mathbb{H}$.

$\forall\, v \in \mathbb{H} = \{\text{quaternions}\} \cong \mathbb{R}^4 \circlearrowleft SO(4)$.

$\forall\, u \in \{\text{unit quaternions}\} \cong S^3$.

$\forall\ v \in \mathbb{H} = \{\text{quaternions}\} \cong \mathbb{R}^4 \circlearrowleft SO(4)$.

$\forall\ u \in \{\text{unit quaternions}\} \cong S^3$.

$\forall\ (h,j) \in \mathbb{Z}^2$.

let • $\xi_{hj}$ be

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r] & E_{hj} \arrow[d] \\ & S^4 \end{tikzcd}$$

oriented vect. bun.

• $M_{hj}$ := unit vectors in $E_{hj}$

$$\begin{tikzcd} S^3 \arrow[r] & M_{hj} \arrow[d] \\ & S^4 \end{tikzcd} \leftarrow \text{smooth 7-manifold.}$$

Example. $(h,j) = (1,0)$.

• $f_{10}$ = standard action $\{\text{Unit quat}\} \curvearrowright \mathbb{H}$

$$\begin{tikzcd} \{\text{Unit quat}\} \arrow[d, phantom, "\cong"] \\ SU(2) \cong S^3 \curvearrowright \mathbb{R}^4. \end{tikzcd}$$

• $\xi_{10}$ :

$$\begin{tikzcd} \mathbb{R}^4 \arrow[r] \arrow[d, phantom, "\cong"] & E_{1,0} \arrow[r] \arrow[d, phantom, "\cong"] & S^4 \arrow[d, phantom, "\cong"] & \\ \mathbb{H}^1 \arrow[r] & E \arrow[r] & \mathbb{HP}^1 = & \dfrac{U(\mathbb{H}^2)}{U(\mathbb{H}^1)} \end{tikzcd}$$

"Hopf bundle"

• $M_{10}$ :

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, phantom, "\cong"] & M_{10} \arrow[r] \arrow[d, phantom, "\cong"] & S^4 \arrow[d, phantom, "\cong"] \\ U(\mathbb{H}^1) \arrow[r] & U(\mathbb{H}^2) \arrow[r] \arrow[d, phantom, "\cong"] & \dfrac{U(\mathbb{H}^2)}{U(\mathbb{H}^1)} \\ S^3 \arrow[r] & S^7 \arrow[r] & S^4 \end{tikzcd}$$

## (II) The invariant $\lambda(M^7)$.

Assume $M^7$ = a closed oriented smooth 7-manifold

s.t. $\boxed{H^3(M^7;\mathbb{Z}) = 0 \quad \text{and} \quad H^4(M^7;\mathbb{Z}) = 0.} \quad (*)$

( later: some $M_{h,j} \cong_{top} S^7$ satisfies $(*)$ ).

Thom's cobordism thm $\Rightarrow \Omega_7 = 0$.

$\Rightarrow \exists\ B^8$ compact, oriented s.t. $\partial B^8 = M^7$.

orientation on $B^8$ $\rightsquigarrow$

$$\begin{tikzcd} {[B]} \arrow[d, mapsto] & \in & H_8(B^8, \partial B^8) \arrow[d, "\partial"] \\ {[M]} & \in & H_7(\partial B^8) \arrow[d, phantom, "="] \\ & & M^7 \end{tikzcd}$$

Poincaré pairing :

$$H^4(B^8, M^7)/_{\text{torsion}} \times H^4(B^8,M^7)/_{\text{torsion}} \xrightarrow{PB} \mathbb{Z}$$

$$(\alpha, \beta) \longmapsto \langle \underset{\text{relative } H^8}{\alpha \cup \beta}, \underset{\text{relative } H_8}{[B]} \rangle$$

• $\sigma(B^8)$ := signature of $PB \in \mathbb{Z}$.

$p_1 := p_1(\tau_{B^8}) \in H^4(B^8)$.

$p_1 := p_1(\tau_{B^8}) \in H^4(B^8)$.

$(*) \Rightarrow \underline{i^* : H^4(B^8, M^7) \xrightarrow{\cong} H^4(B^8)}$ is iso.

Define a "Pontryagin number". $p_1^2$

- $q(B^8) := \langle \underset{H^8(B^8, M^7)}{\underset{\cap}{(i^{*-1} p_1)^2}}, \underset{H_8(B,M)}{\underset{\cap}{[B]}} \rangle \in \mathbb{Z}$.

Thm 1: $2q(B^8) - \sigma(B^8) \in \mathbb{Z}/7\mathbb{Z}$ (mod 7)
does not depend on the choice of $B^8$ s.t. $\partial B^8 = M^7$.

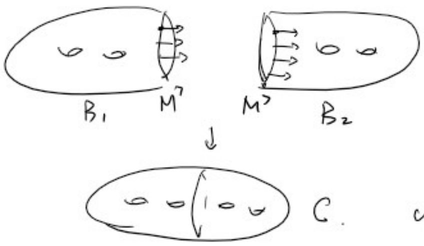
→ (☆) Def: $\lambda(M^7) := 2q(B^8) - \sigma(B^8)$ (mod 7)
$\in \mathbb{Z}/7\mathbb{Z}$. well-defined

→ Cor 2: If $\lambda(M^7) \neq 0$, then $M^7 \underset{\text{diff.}}{\not\approx} S^7$.

Pf. If $M^7 \approx S^7 = \partial B^8$ B = standard 8-ball in $\mathbb{R}^8$.
then $H^4(B^8) = 0 \Rightarrow \begin{cases} \sigma(B) = 0 \\ q(B) = 0 \end{cases} \Rightarrow \lambda(M^7) = 0$. □

pf of Thm1. Suppose $B_1^8$, $B_2^8$ s.t. $\partial B_1 = \partial B_2 = M$.

Consider $C^8 := B_1^8 \cup_{M^7} B_2^8$ closed manifold.



closed 8-manifold.

choose orientation on C that restricts to $B_1$ and $-B_2$.

$q(C^8) = p_1^2[C^8]$

Hirzebruch Signature Thm

$$\Rightarrow \sigma(C^8) = \frac{1}{45}\left(7p_2[C] - \underbrace{p_1^2[C]}_{q[C]}\right)$$

$$\Rightarrow 45\sigma(C^8) + q(C^8) = 7p_2[C] \equiv 0 \pmod 7$$

$\Rightarrow$ in $\mathbb{Z}/7\mathbb{Z}$,

$$0 = 2 \cdot (\underbrace{45\sigma + q}) = 90\sigma + 2q = 2q - \sigma.$$

$$\Rightarrow \underbrace{2q(C^8) - \overset{\overset{\shortparallel}{0}}{\sigma(C^8)}}_{\lambda(C^8)} = 0$$

$$\Rightarrow \lambda(B_1) - \lambda(B_2) = 0. \qquad \square.$$

<u>properties of λ- invariants.</u> $\qquad \lambda = 2q - \sigma \downarrow q = p_1^2$

① $\lambda(-M^7) = -\lambda(M^7)$

② If $\lambda(M^7) \neq 0$, then $\nexists$ orientation - reversing diffeo $M^7 \to M^7$.
$\lambda \neq -\lambda$.

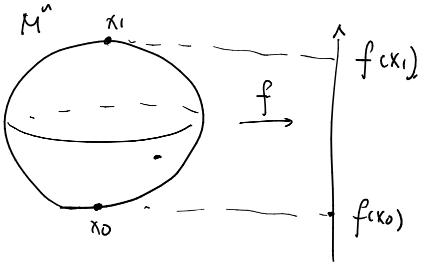
## (III) Topological n-sphere & Morse theory

$M^n$ = closed smooth manifold

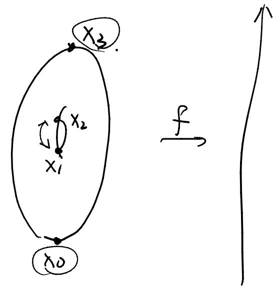
<u>Assumption (H)</u> : $M^n$ admits a differential function $f: M^n \to \mathbb{R}$
having <u>only</u> two critical points $x_0, x_1$.
Moreover, these crit. pts. are <u>nondegenerate</u>.
i.e. in local coordinates $u$

$$H(f) = \left( \frac{\partial^2 f}{\partial u_i \partial u_j} \right)_{i,j} \text{ is nonsingular at } x_0, x_1.$$

<u>pic</u>:



<u>non-Ex</u>:

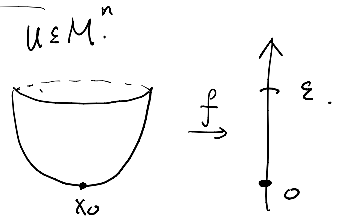


<u>Thm</u> (Reeb). If $M^n$ satisfies (H),
then $\exists$ a <u>homeomorphism</u> $M^n \longrightarrow S^n$
that is a diffeo everywhere except possibly at a single point.

<u>pf sketch</u>: $f: M^n \to \mathbb{R}$ say $f(x_0) = 0$. $f(x_1) = 1$.

<u>Morse lemma</u>: $\exists$ local coordinate $(u_1, \cdots, u_n)$ on a neighborhood $U$ of $x_0 \in M$ s.t.

$$f(x) = \underbrace{u_1^2 + u_2^2 + \cdots + u_n^2}_{U \subset M^n.} \qquad \forall x \in U.$$



Put a Riemannian metric on $U$ by:

$$ds^2 = du_1^2 + \cdots + du_n^2 \leftarrow \text{(standard)}$$

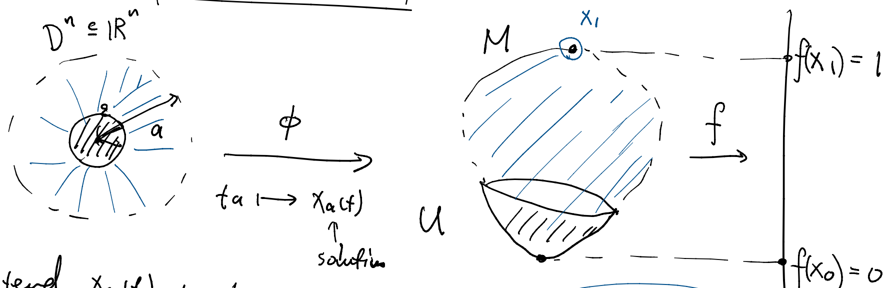
and extend $ds^2$ to entire $M^n$.
$\leftarrow$ Riemanian manifold.

Consider the ODE.

$$\frac{dx(t)}{dt} = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|^2}$$

$\exists$ solutions $x_a(t)$ for $t \in [0,\varepsilon)$ for $a \in \mathbb{R}^n$, $\|a\| = 1$.

then $\boxed{f(x_a(t)) = t}$



Extend $x_a(t)$ to $t \in [0,1]$ satisfying $\boxed{f(x_a(t)) = t.}$

Then the map

$$D^n \xrightarrow{\phi} M^n, \quad ta \longmapsto x_a(t).$$

$f^{-1}[0,1)$

satisfies that

$$\phi\big|_{\operatorname{int}(D^n)} : \operatorname{int}(D^n) = \mathring{D}^n \longrightarrow M^n \setminus \{x_1\}.$$

is a diffeo.

$\bar{\phi}: \mathring{D}^n \cup \{\text{north pole}\} \longrightarrow M^n$ is a homeo. □.

## (IV). Compute invariants of $M_{h,j}$

Consider

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, phantom, "\cap"] & M_{h,j} \arrow[r] \arrow[d, phantom, "\cap"] & S^4 \\ \mathbb{R}^4 \arrow[r] & E_{h,j} \arrow[r] & S^4 \end{tikzcd}$$

$\xi_{h,j}$

$\rightarrow$ Let $\iota := PD(*) \in H^4(S^4;\mathbb{Z})$.

→ Let $\iota := PD(*) \in H^4(S^4;\mathbb{Z})$.

**Lemma 3**: If $\boxed{h+j = \pm 1,}$ then $M_{h,j}$ satisfies assumption $(H)$. ←
Hence, $M_{h,j}$ is homeo to $S^7$.

Rmk: In particular, $M_{h,j}$ satisfies $(*)$. Hence, $\lambda(M_{h,j}) \overset{\in \mathbb{Z}/7\mathbb{Z}.}{\text{ is defined}}$.

**Lemma 4**: $p_1(\xi_{h,j}) = \pm \overset{\iota\ 2\mathbb{Z}}{\overset{\downarrow}{2}}(h-j)\iota \in H^4(S^4)$.

**Lemma 5**: When $h+j=1$, $\lambda(M_{h,j}) = (h-j)^2-1 \in \mathbb{Z}/7\mathbb{Z}$.

Pf. (lemmas ⇒ main Thm).

pick $(h,j)$ s.t. $\begin{cases} h+j=1 \\ (h-j)^2-1 \neq 0 \pmod 7 \end{cases}$

e.g. $(3,-2)$.

Lemma 3 $\Rightarrow$ $M_{h,j}$ homeo to $S^7$.

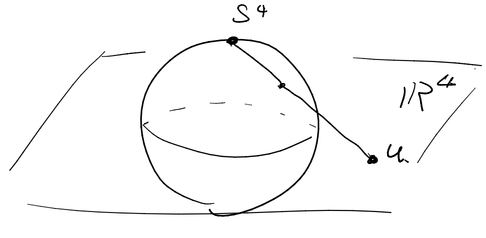
→ Hirzebruch Sig.

Lemma 5 $\Rightarrow$ $\lambda(M_{h,j}) \neq 0$. $\xRightarrow{\text{Cor 2}}$ $(※)$. $M_{h,j}$ is NOT diffeo to $S^7$. □.

pf of lemma 3: Want to produce $f: M_{h,j} \to \mathbb{R}$ satisfying $(H)$
i.e. $f$ has only 2 critical pts.

$S^3 \to M \xrightarrow{\pi} S^4$

Consider base $S^4$, choose charts on $S^4$:



$u : \mathbb{R}^4 \longrightarrow S^4 \setminus \{\text{north pole}\}$.

$u' : \mathbb{R}^4 \longrightarrow S^4 \setminus \{\text{south p..}\}$.

transition funtion:

$$u' = \frac{u}{\|u\|^2} : \mathbb{R}^4 \setminus 0 \xrightarrow{\cong} \mathbb{R}^4 \setminus 0$$

To construct the total space $M^7 = M_{h,j}$,

take 2 copies of $\mathbb{R}^4 \times S^3$ and identify by:

$$(\mathbb{R}^4 \setminus 0) \times S^3 \xrightarrow[\cong]{\phi} (\mathbb{R}^4 \setminus 0) \times S^3$$

$$\xrightarrow{\cong} \quad (\mathbb{R} \setminus 0) \times S$$

$$(u, v) \longmapsto (u', v') := \left( \frac{u}{\|u\|^2}, \frac{u^h v u^j}{\|u\|} \right)$$

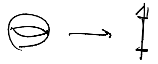
$\xrightarrow{f} \mathbb{R}$ $\quad \overset{\cap}{\mathbb{H}} \quad \overset{\cap}{\mathbb{H}}$ $\quad \uparrow$ quaternions.

$M_{h,j}$ is a smooth manifold.

change coordinates

$(u', v')$ to $(u'', v')$ where $u'' = u' \cdot (v')^{-1}$

Define $f : M_{h,j} \longrightarrow \mathbb{R}$ on 2 charts:



$v \in S^3 \longrightarrow [-1,1]$

$$\rightarrow f(u,v) := \frac{Re(v)}{\sqrt{1 + \|u\|^2}} \qquad \left( \begin{matrix} \cap \\ Re : \mathbb{H} \longrightarrow \mathbb{R} \\ \text{real part} \end{matrix} \right)$$

$\searrow$ crit pt. only when $\|u\| = 0$. or $u = 0$.

$$f(u'', v') := \frac{Re(u'')}{\sqrt{1 + \|u''\|^2}}$$

(you check): $h + j = -1 \Rightarrow f$ agrees on the intersection of 2 charts.

$\left( h + j = +1, \text{ note: } \overline{\xi_{h,j}} \cong \xi_{-j,-h} \right)$

Critical pts:

on $(u,v)$-chart, fix $u$. critical pt occur if $Re(v) = \pm 1$.

2 critical pts are $(u,v) = (0, \pm 1)$.

note: $\frac{d}{dt}\left( \frac{1}{\sqrt{1+t^2}} \right) = \frac{-2t}{\sqrt{1+t^2}} \neq 0$ if $t \neq 0$.

You check: no critical pt on $(u'', v')$ chart. $\square$.

pf of Lemma 4:

$$\mathbb{Z}^2 \Rightarrow \pi_3 SO(4) \cong \pi_4 BSO(4) \cong [S^4, BSO(4)] \xrightarrow{p_1} H^4(S^4) \cong \mathbb{Z}\{\iota\}.$$

group homo.

$$(h,j) \longmapsto \xi_{h,j} \longrightarrow p_1(\xi_{h,j})$$

So $\begin{matrix} \mathbb{Z}^2 \longrightarrow \mathbb{Z} \\ (h,j) \longmapsto p_1(\xi_{h,j}) \end{matrix}$ is $\mathbb{Z}$-linear.

$\iota = PD(*) \in H^4(S^4)$

$\Rightarrow \exists\, c, d \in \mathbb{Z}$ s.t. $p_1(\xi_{h,j}) = (ch + dj)\, \iota$

$\Rightarrow \exists\, c, d \in \mathbb{Z}$ s.t. $p_1(\xi_{h,j}) = (ch + dj)\,\iota$ ✓

note: $\xi_{-j,-h} \cong \overline{\xi_{h,j}}$

$$\Rightarrow \begin{array}{ccc} p_1(\xi_{h,j}) & = & p_1(\xi_{-j,-h}) \\ \| & & \| \\ ch+dj & = & c(-j)+d(-h) \end{array}$$

$\Rightarrow (c+d)(h-j) = 0 \quad \forall\, h,j \in \mathbb{Z}.$

$\Rightarrow c = -d.$

thus, $p_1(\xi_{h,j}) = c(h-j)\,\iota$ for some $c \in \mathbb{Z}$.

we will show $c = \pm 2$ in the next pf. ✓

pf of Lemma 5: Let $M := M_{h,j}$. $(h+j = +1.)$

$$\begin{tikzcd} S^3 \arrow[r] \arrow[d, phantom, "\cap"] & M^7 \arrow[r] \arrow[d, phantom, "\cap"] & S^4 \arrow[d, phantom, "\|"] \\ D^4 \arrow[r] \arrow[d, phantom, "\cap"] & B^8 \arrow[r] & S^4 \arrow[d, phantom, "\|"] \\ \mathbb{R}^4 \arrow[r] & E \arrow[r] & S^4 \end{tikzcd}$$

$\lambda = 2q - \sigma$

$q = p_1^2$

vector $\|\cdot\| \le 1$ in $\xi_{h,j}$.

$\xi_{h,j}$

✓ Compute $\sigma(B)$: $H^4(B) \cong H^4(S^4)$

$$\alpha \mathrel{\reflectbox{$\longmapsto$}} \iota$$

$$P_B(\alpha,\alpha) = \left\langle \underset{H^{8}(B,M)}{(i^{-1}\alpha)^2}, \underset{H_8}{[B^8]} \right\rangle = \pm 1.$$

$\uparrow$

choose orientation on $B^8$ s.t. $= +1$.

$\Rightarrow \sigma(B^8) = +1.$

Compute $q(B) = \langle (i^{-1} p_1(\tau_B))^2, [B^8] \rangle.$

Decompose tangent bundle $\tau_B := \eta^4 \oplus \nu^4$

$D^4 \to D \xrightarrow{\pi} S^4$

Decompose tangent bundle $D^4 \to B \xrightarrow{\pi} S^4$

$$\begin{tikzcd} & \tau_B := \eta^4 \arrow[d] \oplus \nu^4 \arrow[dr] & \\ & \text{tangent to fibers} \cong D^4 & \text{normal to fibers. (tangent to } S^4\text{).} \end{tikzcd}$$

$\eta$:

$$\begin{tikzcd} \eta^4 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \xi^4 = \xi_{h,j} \arrow[d] \\ B^8 \arrow[r, "\pi"] & S^4 \end{tikzcd}$$

$$\begin{tikzcd} H^4(B^8) & H^4(S^4) \arrow[l, "\pi^*", "\cong"'] \\ p_1(\eta) & p_1(\xi_{h,j}) \arrow[l, mapsto] \\ c(h-j)\alpha & c(h-j)\iota \arrow[l, mapsto] \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

$\nu$:

$$\begin{tikzcd} \nu^4 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tau_{S^4} \arrow[d] \\ B^8 \arrow[r, "\pi"] & S^4 \end{tikzcd}$$

$$\begin{tikzcd} H^4(B^8) & H^4(S^4) \arrow[l, "\pi^*"] \\ p_1(\nu) \arrow[d, phantom, "\shortparallel"] & p_1(\tau_{S^4}) \arrow[l, mapsto] \arrow[d, phantom, "\shortparallel"] \\ 0. & 0 \text{ (before).} \end{tikzcd}$$

Thus, $p_1(\tau_B) = p_1(\eta \oplus \nu) = p_1(\eta) + p_1(\nu) = \boxed{c(h-j)\alpha + 0.}$ $\forall h,j$.

To compute $c \in \mathbb{Z}$, consider $(h,j) = (1,0)$

$$\xi_{1,0} = \text{canonical} \quad \begin{tikzcd} \mathbb{H} = \mathbb{R}^4 \arrow[r] & E \arrow[d] \\ & \mathbb{H}P^1 = S^4 \end{tikzcd}$$ "Hopf bundle".

explicit calculation. $p_1(\tau_{B^8}) = 2 \cdot \text{generator} \in H^4(S^4)$

$\Rightarrow c = \pm 2.$

Finally. $$q(B^8) = \langle [\underbrace{i^{-1} p_1(B^8)}_{\shortparallel \atop \pm 2(h-j)\alpha}]^2, [B] \rangle$$

$$= 4(h-j)^2 \underbrace{\langle i^{-1}(\alpha)^2, [B] \rangle}_{\text{chosen to be } \pm 1.}$$

$\lambda(B) = 2q(B) - \sigma(B)$

$-1.$

$$\begin{aligned}\lambda(B) &= 2q(B) - \sigma(B) \\ &= 8(h-j)^2 - 1 \\ &= (h-j)^2 - 1 \quad \in \mathbb{Z}/7\mathbb{Z}.\end{aligned}$$

$\square$.