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目录

Part I

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Chapter 1

Elliptic functions

1 Simply periodic functions

Notations

- region=domain=open connected subset of $\mathbb{C} = \{x + \mathrm{i}y : x, y \in \mathbb{C}\}$ endowed the Euclidean topology
- Let $\Omega \xrightarrow{f} \overline{\mathbb{C}}$ be meromorphic,where Ω is a region.Assume that Ω is left invariant under the translation $z \mapsto z + \omega$,where $\omega \in \mathbb{C}^* = C\backslash\{0\}$

  Suppose that $f(z + \omega) = f(z), \forall z \in \Omega$i.e. f has period ω. Then $n\omega(n \in \mathbb{Z})$ are also periods of f.

  Call f a simply periodic function on Ω

Example 1.1. $\Omega = \mathbb{C}$,e^z *has period* $2\pi\mathrm{i}$,$\cos z$ *and* $\sin z$ *have period* 2π*.*

1.1 Representation by exp

Define $\Omega' = \left\{\zeta \in \mathbb{C} : \zeta = \mathrm{e}^{\frac{2\pi\mathrm{i}z}{\omega}}, z \in \Omega\right\}$

Example 1.2.

- $\mathbb{C} \xrightarrow{\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega}}} \mathbb{C}^*$
- $\Omega = \left\{a < \Im\dfrac{2\pi z}{\omega} < b\right\} \xrightarrow{\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega}}} \Omega' = \left\{\mathrm{e}^{-b} < |\zeta| < \mathrm{e}^{-a}\right\}$

Observation: Use notations as above,$\exists$ a unique mero fun $\Omega' \xrightarrow{F} \overline{\mathbb{C}}$ s.t.

(1) $f(z) = F(\mathrm{e}^{\frac{2\pi\mathrm{i}z}{\omega}})$

...

Take $\zeta \in \Omega'$,$\exists$ "unique" $z \in \Omega$ up to translation s.t. $\zeta = \mathrm{e}^{\frac{2\pi\mathrm{i}z}{\omega}}$ It follows from that ω is period of f

Conversely,giver a mero fun $F : \Omega' \longrightarrow \overline{\mathbb{C}}$, we obtain by (1) a periodic mero func $\Omega \xrightarrow{f} \overline{\mathbb{C}}$

1.2 Fourier development

Assumption: Suppose that F is holomorphic in annulus $\{r_1 < |\zeta| < r_2\}\,(0 \leqslant r_1 < r_2 \leqslant +\infty)$ Then F has its Laurent development in the annulus,

$$F(\zeta) = \sum_{n=-\infty}^{+\infty} c_n\zeta^n, c_n = \frac{1}{2\pi\mathrm{i}}\int_{|\zeta|=r} F(\zeta\zeta^{-n-1})\mathrm{d}\zeta(r_1 < r < r_2)$$

Hence we obtain the complex Fourier development of $f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n\mathrm{e}^{\frac{2\pi\mathrm{i}z}{\omega}}, c_n = \frac{1}{\omega}\int_a^{a+\omega} f(z)\mathrm{e}^{\frac{-2\pi inz}{\omega}}\mathrm{d}z$$

in the corresponding strip $\left\{-\ln r_2 < \Im\frac{2\pi z}{\omega} < -\ln r_1\right\}$

Example 1.3. *As* $\Omega = \mathbb{C}, \Omega' = \mathbb{C}^*$, *the complex Fourier development of* f *holds everywhere.*

1.3 Functions of finite order

Let $\Omega = \mathbb{C}$ and $\Omega' = \mathbb{C}^*$, $F : \Omega' \overset{mero}{\longrightarrow} \overline{\mathbb{C}}$ has at most poles at $0, \infty$

Then F is rational,i.e.

$$F = \frac{polynomial}{polynomial}$$

with degree d. We say f is finite order, equal to $d = \deg F$.

Define a congruent relation $z \sim z + \omega$,which is an equivalence relation on $\mathbb{C}$

The set of congruent classes can be identified with the periodic strip$S = \left\{0 \leqslant \Im\frac{2\pi z}{\omega} < 2\pi\right\}$

By the commutative diagram

$$\begin{array}{ccc}\mathbb{C} & \xrightarrow{f} & \overline{\mathbb{C}} \\ \downarrow & & \\ \mathbb{C}^* & & \end{array}$$

f is of oerder d, assumes each $c \in \mathbb{C}\setminus\{F(0), F(\infty)\}$

at d different congruent classes.

Since $f(z) \to F(0)$ as $\Im\frac{z}{\omega} \to -\infty$; $f(z) \to F(\infty)$ as $\operatorname{Im}\frac{z}{\omega} \to +\infty$

we can understand that f assums both $F(0)$ and $F(\infty)$ with multiplicity d.

Since the strip S contains only one representative of each congruent class, f assumes each $c \in \overline{\mathbb{C}}$ d times in S,with a special case for $F(0)$ and $F(\infty)$.

2 Doubly perodic functions

Definition 2.1. *Elliptic functions are mero functions with two $\mathbb{R}$-linear independent periods on $\mathbb{C}$.*

2.1 The period module($\mathbb{Z}$ module)

Let $\mathbb{C} \xrightarrow{f} \overline{C}$ be mero and M the set of periods of f.

M may be $\{0\}$. If $\omega_1, \omega_2 \in M$ then $n_1\omega_1 + n_2\omega_2 \in M \forall n_1, n_2 \in \mathbb{Z}$. Hence M is a $\mathbb{Z}$-module

观察：假设 f 不是常值则 M 离散

Call M the period module of f.

Theorem 2.1 (Classification of period modules)**.** *Assume f non const.Then M can be classified as:* $\{0\}$,$\mathbb{Z}\omega(\omega \in \mathbb{C}^*)$,$\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2(\dfrac{\omega_2}{\omega_1} \notin \mathbb{R})$

证明. Assume $M \neq \{0\}$.

由离散性，可取一个以原点为圆心，r 为半径的闭圆盘，使得其中有有限个 M 中元素。

Since M is discrete,$\exists 0 \neq \omega_1 \in M$ s.t. $|\omega_1| = \infty_{\omega \neq 0}|\omega|$

Assume $M \supsetneq \mathbb{Z}\omega_1$ Take $\omega_2 \in M \backslash \mathbb{Z}\omega_1$ s.t.

$$|\omega_2| = \infty_{\omega \in M \backslash \mathbb{Z}\omega_1}|\omega|$$

Then $\dfrac{\omega_2}{\omega_1} \notin \mathbb{R}$,otherwies, $\exists n \in \mathbb{Z}$,$n < \dfrac{\omega_2}{\omega_1} < n+1$. Then $0 < |n\omega_1 - \omega_2| < |\omega_1|$, contradict the definition of ω_1

The problem is reduced to the following claim

CLAIM: $M = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

Since $\dfrac{\omega_1}{\omega_1} \notin \mathbb{R}$,$\begin{vmatrix} \omega_1 & \omega_2 \\ \bar{\omega_1} & \bar{\omega_2} \end{vmatrix}$

$\forall \omega \in C$,solving the equations

$$\begin{cases} \omega = \lambda_1\omega_1 + \lambda_2\omega_2 \\ \bar{\omega} = \lambda_1\bar{\omega_1} + \lambda_2\bar{\omega_2} \end{cases}$$

we find$\lambda_1, \lambda_2 \in \mathbb{C}$

Choose $m_1, m_2 \in \mathbb{Z} : |\lambda_1 - m_1|, |\lambda_2 - m_2| \leqslant \dfrac{1}{2}$

Assume further $\omega \in M$,Setting $\omega' = \omega - m_1\omega_1 - m_2\omega_2$,we have

$$|\omega'| = |(\lambda_1 - m_1)\omega_1 + (\lambda_2 - m_2)\omega_2| < |\lambda_1 - m_1||\omega_1| + |\lambda_2 - m_2||\omega_2| \leqslant \frac{1}{2}(|\omega_1| + |\omega_2|) \leqslant |\omega_2|$$

By the definition of ω_2,since $\omega' \in M$, $\omega' \in \mathbb{Z}\omega_1$ □

From now on we assume $M = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ is the period module of an elliptic function $f : \mathbb{C} \longrightarrow \overline{\mathbb{C}}$

2.2 Unimodular transform

模群，

- $GL(2, \mathbb{Z})$，阿尔福斯

- $SL(2,\mathbb{Z})$，维基
- $PSL(2,\mathbb{Z})$，维基

Call a pair $(\omega_1' - \omega_2')$ a basis of M if $M = \mathbb{Z}\omega_1' \oplus \mathbb{Z}\omega_2'$

Relation between two bases (ω_1, ω_2) and (ω_1', ω_2') of M

choose $a, b, c, d \in \mathbb{Z}$ s.t.

$$(2)\begin{cases} \omega_2' = a\omega_2 + b\omega_1 \\ \omega_1' = c\omega_2 + d\omega_1 \end{cases} \quad \begin{cases} \omega_2 = a'\omega_2' + b'\omega_1' \\ \omega_1 = c'\omega_2' + d'\omega_1' \end{cases}$$

use elementary linear algbra, we know htat

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Call a linear transfomr in (2) with integral coefficients and det ± 1 unimodular.

Fact:Any two bases of the same module $M = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega$ are connected by a unimodular transofrm

Notations Modular gp:=$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = \pm 1, a, b, c, d \in \mathbb{Z} \right\}$

Denote $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$,

$$PSL(2, R) = \left\{ Mbiustransformz \mapsto \frac{az + b}{cz + d} \mid a, b, c, d \in R, ad - bc = 1 \right\}$$

Example 2.1. $PSL(2,\mathbb{R}), PSL(2,\mathbb{Z}) \frown \mathcal{H} = \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

2.3 The canonical basis

In the proof of Theorem 1 we roughly obtained a cononical basis (ω_2, ω_1) s.t.

$$\frac{\omega_2}{\omega_1} \in thefundamentalregioninFig1.$$

Theorem 2.2. *$\exists$ a basis (ω_1, ω_2) of M s.t. $\tau = \frac{\omega_2}{\omega_1}$ satisfieds the followint conditions*

(i) $\Im\tau > 0$

(ii) $-\frac{1}{2} < \Re\tau \leqslant \frac{1}{2}$

(iii) $|\tau| \geqslant 1$

(iv) $\Re\tau \geqslant 0$ *if* $|\tau| = 1$

The ratio τ is uniquely determined by these conditions, and there exists $2, 4$ of 6 choices of canonical bases.

证明. Select ω_1 and ω_2 as in the proof of Theorem1 such that

$$|\omega_1| \leqslant |\omega_2|, |\omega_2| \leqslant |\omega_1+\omega_2| \text{ and } |\omega_2| \leqslant |\omega_1-\omega_2|,$$

which are equivalent to

$$1 \leqslant |\tau|, |\tau| \leqslant |1+\tau| \text{ and } |\tau| \leqslant |1-\tau|.$$

Take another canonical basis $(\omega_2', \omega_1')^T$ satisfy

$$\begin{pmatrix}\omega_2' \\ \omega_1'\end{pmatrix} = \begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}\omega_2 \\ \omega_1\end{pmatrix}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \pm 1$.

$\tau' = \dfrac{a\tau+b}{c\tau+d} = \dfrac{ac|\tau|^2 + bd + (ad+bc)\Re\tau + \mathrm{i}(ad-bc)\Im\tau}{|c\tau+d|^2}, \Im\tau' = \dfrac{(ad-bc)\Im\tau}{|c\tau+d|^2}$

(i) $\Longrightarrow ad - bc = 1$

We may assume that $\Im\tau' \geqslant \Im\tau$, then $|c\tau + d| \leqslant 1$.

Case1 $c = 0$

$|d| \leqslant 1$ and $d \in \mathbb{Z} \Longrightarrow d = \pm 1$

$ad - bc = 1 \Longrightarrow a = d = \pm 1$

$\Longrightarrow \tau' = \dfrac{a\tau + b}{d} = \tau \pm b$

$\Longrightarrow \Re\tau' - \Re\tau = \pm b \in \mathbb{Z}, \Im\tau' = \Im\tau$

(ii)$\Re\tau - \Re\tau' \in (-1, 1) \Longrightarrow |b| < 1 \Longrightarrow b = 0 \Longrightarrow \tau' = \tau$

$\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}$

Case2 Assume that $c \neq 0$ from now on ,tehn $|c| = 1$

If $|c| \geqslant 2, \left|\tau + \dfrac{d}{c}\right| \leqslant \dfrac{1}{|c|} \leqslant \dfrac{1}{2}$, which is a contradiction.

Hence

$$\begin{cases} c = 1, |\tau + d| \leqslant 1 \\ c = -1, |\tau - 1| \leqslant 1 \end{cases} \Longrightarrow \begin{cases} c = \pm 1, d = 0, |\tau| = 1 \\ d = -c = \pm 1, \tau = \mathrm{e}^{\frac{\pi \mathrm{i}}{3}} \end{cases}$$

(1) $|\tau| = 1, d = 0, c = \pm 1$

$|c\tau + d| = |\tau| = 1 \Longrightarrow \Im\tau' = \Im\tau$

$ad - bc = 1 \Longrightarrow bc = -1 \Longrightarrow b = -c = \pm 1$

$\tau' = \dfrac{a\tau + b}{c\tau} = \dfrac{a}{c} + \dfrac{b}{c}\cdot\dfrac{1}{c} = \dfrac{a}{c} - \dfrac{1}{\tau} = \pm a - \dfrac{1}{\tau} = \pm a - \Re\tau + \mathrm{i}\Im\tau$

$\Longrightarrow \Re\tau + \Re\tau' = \pm a \in \mathbb{Z}$

$\Re\tau + \Re\tau' \in (-1, 1]$

- When $a = 0, \Re\tau' = -\Re\tau$

  $|\tau| = 1 \overset{\text{(iv)}}{\Longrightarrow} \Re\tau = 0$

  $\tau' = \tau = \mathrm{i}\Im\tau \overset{|\tau|=1}{\Longrightarrow} \Im\tau = 1, \tau' = \tau = \mathrm{i}$

  $\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$

- When $a = \pm 1$, $\Re\tau = \Re\tau' = \dfrac{1}{2} \Longrightarrow \tau' = \tau = \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}$
  $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$

(2) $\tau = \mathrm{e}^{\frac{\pi\mathrm{i}}{3}}, d = -c = \pm 1$

$|c\tau + d| = 1 \Longrightarrow \Im\tau' = \Im\tau$

$ad - bc = 1 \Longrightarrow a + b = d$

$$\tau' = \begin{cases} \dfrac{1}{2}(1-2a) + \mathrm{i}\dfrac{\sqrt{3}}{2}, d = -c = 1 \\ \dfrac{1}{2}(1+2a) + \mathrm{i}\dfrac{\sqrt{3}}{2}, d = -c = -1 \end{cases}$$

$$\begin{cases} a = 0, d = -c = 1, b = 1 \\ a = 0, b = -1, d = -c = -1 \end{cases}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

□

2.4 General properties of elliptic functions

Suppose that $f : \mathbb{C} \to \overline{C}$ is meromorphic with preiod module $M = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 (\dfrac{\omega_2}{\omega_2} \notin \mathbb{R})$.Then f takes the same value at each congruent class where we say that $z_1 \equiv z_2 (\mod M)$ iff $z_1 - z_2 \in M$.

$\forall a \in \mathbb{C}$, set $P_a = \{a + t_1\omega_1 + t_2\omega_2 : 0 \leqslant t_1, t_2 \leqslant 1\}$, f is completely determined by its values in P_a.

Assume that all elliptic functions are non-constant if otherwise stated.

Observation:$\exists a \in C$ s.t. f has neither poles nor zeros on ∂P_a.

证明. P_f:pole set,Z_f:zero set

$L_s = \{a + s\omega_1 + t\omega_2, 0 \leqslant t \leqslant 1\}$

$L_{s_1} \cap L_{s_2} = \varnothing, \forall s_1, s_2 \in [0, 1]$

$\exists s_0 \in [0, 1], L_{s_0} \cap P_f = \varnothing$ and $L_{s_0} \cap Z_f = \varnothing$

$L'_t = \{a + t\omega_2 + s\omega_1 : 0 \leqslant s \leqslant 1\}$

$\exists t_0 \in [0, 1], L'_{t_0} \cap P_f = L'_{t_0} \cap Z_f = \varnothing$.

Set $b = L_{s_0} \cap L'_{t_0}$. □

Theorem 2.3. *Each non constant elliptic function has poles.*

Remark. *A pole of an elliptic function means a congruent class. Then an elliptic function has finitely many poles. We count the order of a pole as the usual way.*

Theorem 2.4. *The sum of the residues of the poles of an elliptic function vanishes.*

证明. We take ∂P_a as bellow, where f has no pole on ∂P_a. By the residue theorem,

$$\sum_{z \in Int P_a} residue of pole z = \int_{\partial P_a} f(z)\mathrm{d}z = 0$$

□

Remark. *There exists no elliptic function with a single simple pole.*

Theorem 2.5. *A nonconst elliptic function has equally many poles as its zeros.*

证明. Observe that $\frac{f'(z)}{f(z)}$ is elliptic and poles and zeros of f are simple poles of $\frac{f'(z)}{f(z)}$. Moreever, we have

$$Res_P \frac{f'(z)}{f(z)} = \begin{cases} mult_P(f), & P \text{is a zero of} f \\ -ord_P(f), & P \text{is a pole of} f \end{cases}$$

Then

$$0 = \int_{\partial P_a} \frac{f'(z)}{f(z)} \mathrm{d}z = \text{number of zeros of } f - \text{number of poles of} f$$

□

Definition 2.2. *$\forall f \in \mathbb{C}$, $f(z) - c$ has the same poles as f. Hence, all complex numbers are assumed equally many times by f. We call the number of in congruent roots of equation $f(z) - c = 0$ the order of f.*

Theorem 2.6. *The zeros $a_1, \cdots, a_n$ and poles $b_1, \cdots, b_n$ of an elliptic function of order n satisfy*

$$a_1 + \cdots + a_n \equiv b_1 + \cdots + b_n (\mod M).$$

f even elliptic function with period ω_1, ω_2, can be expressed in the form

$$C \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

provided that 0 is neither a zero nor a pole.

证明. $g(z) = \wp(z) - \wp(u), u \in P_0 \backslash \{0\}$

pole: double pole 0

zero:

- two simple zeros
- one double zero

Let $g(u) = 0$,$u \in Z_f \Longrightarrow g(-u)$

$u \equiv -u \mod M \Longrightarrow 2u = m\omega_n\omega_2$ for some $m, n \in \mathbb{Z}$

$u \in P_0 \backslash \{0\} \Longrightarrow u = \frac{\omega_1}{2}, \frac{\omega_2}{2}$ or $\frac{\omega_1 + \omega_2}{2}$

- $u \neq \frac{\omega_1}{2}, \frac{\omega_2}{2}$ and $\frac{\omega_1 + \omega_2}{2} \Longrightarrow$ two simple zeros are $u, -u$
  $L_1 = \{t\omega_1 : t \in [0, 1)\}, L_2 = \{t\omega_2 : t \in [0, 1)\}$
  - When $u \in L_1, -u \equiv \omega_1 - u \mod M$
  - When $u \in L_2, -u \equiv \omega_2 - u \mod M$
  - $u \notin L_1 \cup L_2, -u \equiv \omega_1 + \omega_2 - u \mod M$

- $u = \frac{\omega_1}{2}, \frac{\omega_2}{2}$ or $\frac{\omega_1 + \omega_2}{2}$

  $g(z) = g(-z) \Longrightarrow g^{(2k-1)}(z) = -g^{(2k-1)}(-z) \Longrightarrow g^{(2k-1)}(u) = 0, \forall\ k \geqslant 1$

  $\Longrightarrow u$ is a double zero.

  $a \in Z_f, -a \in Z_f$

  $$\Longrightarrow \begin{cases} a \not\equiv -a \mod M \\ a \equiv -a \mod M \end{cases} \Longrightarrow \begin{cases} 2\text{zeros} : a, -a \\ order(a)\text{is even} \end{cases}$$

  $Z_f = a_1, a_2, \cdots, a_k, -a_1, \cdots, -a_k, 2a_{k+1}, \cdots, 2a_n, a_i \not\equiv -a_i, \forall\ 1 \leqslant i \leqslant k, a_i \equiv -a_i k < i \leqslant n$

  $P_f = b_1, b_2, \cdots, b_l, -b_1, -\cdots, -b_l, 2b_{l+1}, \cdots, 2b_n, b_i \not\equiv -b_i, 1 \leqslant i \leqslant l, b_i \equiv -b_i, l < i \leqslant n$

  $$f = C \prod_{k=1}^{n} \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

□

3 The Weierstrass theory

3.1 $\wp$-function

Want to cunstruct an elliptic function f of order 2 s.t. its Laurent development has form at the origin

$$z^{-2} + 0 + a_1 z + a_2 z^2 + \cdots$$

CLAIM: $f(z) = f(-z), \forall z \in C$ i.e. f is even.Since $f(z) - f(-z)$ is elliptic and holomorhic, $f(z) - f(-z) = const$. On the other hand, $f\left(\frac{\omega_1}{2}\right) - f(-\frac{\omega_1}{2}) = 0$.

Fact(Weierstrass) An elliptic function of order 2 and with principal sigular part z^{-2} at the origin must have form

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left[\frac{1}{(z-\omega^2) - \frac{1}{\omega^2}} \right]$$

证明.

- Uniform convergence on each compact subset of $\mathbb{C}\backslash M$ can be reduced to $\sum_{\omega \neq 0} \frac{1}{|\omega|^3} < +\infty$.
- Denote by f the RHS.

  Termwisely differtntiating, we find

  $$f'(z) = -2 \sum_{\omega \in M} \frac{1}{(z-\omega)^3}$$

  has periods in M. Hence $f(z + w_j) - f(z) \equiv c_j$.

  Choose $z_j = -\frac{\omega_j}{2}$, we have $c_j = f\left(\frac{\omega_j}{2}\right) - f\left(-\frac{\omega_j}{2}\right) = 0$

  Since $\wp(z)$ and $f(z)$ have order 2 and the same principal singular part, $\wp(z) - f(z) = const$

  Therefore $f(z) = \wp(z)$.

□

3.2 The function $\zeta(z)$ and $\sigma(z)$

We have the anti-derivative $-\zeta(z)$ of $\wp(z)$ as

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

Obeservation:$\exists$ constants η_1, η_2 such that

$$\zeta(z + \omega_j) = \zeta(z) + \eta_j, j = 1, 2.$$

Legendre's relation: $\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi \mathrm{i}$

证明. □

Since residues of ζ equal 1, it has no single valued anti-derivative. To eliminate such multiple-valuedness, consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\log\sigma(z) = \frac{\sigma'(z)}{\sigma(z)} = \zeta(z).$$

Observe

计算 $\wp(z), \wp'(z), \wp'(z)^2$,

$$\wp'(z)^2 = 4\wp^3(z) - 60G_2\wp(z) - 140G_3 =: \tag{15}$$

That is, $w = \wp(z)$ satisfies the ODE $\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 = 4w^3 - g_2w - g_3$. Then $\frac{\mathrm{d}z}{\mathrm{d}w} = \frac{1}{\frac{\mathrm{d}w}{\mathrm{d}z}} = \frac{1}{\sqrt{4w^3 - g_2w - g_3}}$

$$z = \int^w \frac{\mathrm{d}w}{\sqrt{4w^3 - g_2w - g_3}} + C$$

Precisely,$z - z_0 = \int_{w_0=\wp(z_0)}^{w=\wp(z)} \frac{\mathrm{d}w}{\sqrt{4w^3 - g_2w - g_3}}$,where the path of integration from $\wp(z_0)$ to $\wp(z)$ is the image under $\wp$ of another path form z_0 to z avoiding both zeros and poles of $\wp'(z)$, and where the sign of square root is chosen so that it equals $\wp'(z)$.

3.3 The modular function $\lambda(\tau)$

Determine the zeros of $\wp'(z)$

Let e_1, e_2, e_3 be the three zeros of polynomial $4w^3 - g_2w - g_3$.

Then we have an alternative expression of (15)

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) \tag{20}$$

Differen

Since $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ are mutually incongruent, they are exactly the three simple zeros of $\wp'(z)$.

We define

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), e_2 = \wp\left(\frac{\omega_2}{2}\right), e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right). \tag{1.1}$$

CLAIM:e_1, e_2, e_3 are mutually distinct.

Since $\wp'(\frac{\omega_1}{2})$,$\wp(z)$ assume e_1 at least twice. If two of them coincided with each other, that value would be assumed $\geqslant$ four times. Contradict with that $\wp$ is of order 2.

Definition of the modular function

$\mathcal{H} = \{\tau \in \mathbb{C} \mid \Im\tau > 0\} \xrightarrow{\lambda} \mathbb{C}\setminus\{0, 1\}$.

Denote $\wp(z)$ by $\wp_{(\omega_1,\omega_2)}(z)$ in order to express its depedence on $M = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$.

Similarly,we use notations $e_{k,(\omega_1,\omega_2)}$ for $k = 1, 2, 3$.

Then, it is easy to check by (9) that

$$e_{k,(t\omega_1,t\omega_2)} = t^{-2}e_{k,(\omega_1,\omega_2)}, \quad t \in \mathbb{C}^\times.$$

Hence we obtain that

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2} = \frac{\wp\left(\frac{\omega_1+\omega_2}{2}\right) - \wp\left(\frac{\omega_2}{2}\right)}{\wp\left(\frac{\omega_1}{2}\right) - \wp\left(\frac{\omega}{2}\right)} \tag{1.2}$$

depend only on $\tau = \dfrac{\omega_2}{\omega_1}$.

$\lambda : \mathcal{H} \to \mathbb{C}\setminus\{0,1\}$ is analytic.

Elliptic modular function

Given a unimodular transform $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\Im\frac{a\tau + b}{c\tau + d} = sgn(ad - bc)\frac{\Im\tau}{|c\tau + d|^2} = \frac{\pm\Im\tau}{|c\tau + d|}.$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ does not preserve $\mathcal{H}$ in general.

Consider the subgroup $\Gamma := SL(2,\mathbb{Z})$ of the modular group.

Define the congruence subgroup mod 2 of $\Gamma = SL(2,\mathbb{Z})$ to be

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}.$$

Each unimodular transform in Γ preserves the period module, but permutes the three half period and then also permutes e_1, e_2, e_3.

However, $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv$

In this sense, $\lambda : \mathcal{H} \to \mathbb{C}$ is called an elliptic modular function.

3.4 The conformal mapping by $\lambda(\tau)$

We normalize $\omega_1 = 1, \omega_2 = \tau \in \mathscr{H}$. We obtain by (9) and (21)

$$e_3 - e_2 = \sum_{m,n=-\infty}^{\infty} \left(\frac{1}{(m - \frac{1}{2} + (n + \frac{1}{2})\tau)^2} - \frac{1}{(m + (n - \frac{1}{2})\tau)^2} \right) \tag{1.3}$$

$$e_1 - e_2 = \sum_{m,n=-\infty}^{\infty} \left(\frac{1}{(m - \frac{1}{2} + n\tau)^2} - \frac{1}{(m + (n - \frac{1}{2})\tau)^2} \right)$$

where the double series absolutely converges and uniformly

Let $\Omega' = \{\tau - 1 \colon \tau \in \Omega\}, \forall \tau' \in \Omega', \tau' + 1 \in \Omega, \lambda(\tau') = \dfrac{\lambda(\tau'+1)}{\lambda(\tau'+1)-1}$. Then λ maps Ω' onto the lower half plane and maps $\overline{\Omega} \cup \Omega'$ onto $\mathbb{C} \setminus \{0,1\}$ (closure taken wrt $\{z \in \mathbb{C} \mid \Im\tau > 0\}$)

Theorem 3.1. *Each $\tau \in \mathscr{H}$ is equivalent to exactly one pt in $\overline{\Omega} \cup \Omega'$ mod $\Gamma(2)$.*

证明. Each unimodular matrix in $SL_2(\mathbb{Z}) = \Gamma$ is congruent mod 2 to one of the following six matrices in Γ

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

denoted by $S_k^{-1}(1 \leqslant k \leqslant 6)$ which acts on $\mathscr{H}$ as Möbius transform. That is,

$$S_1(\tau) = \tau, S_2(\tau) = -\frac{1}{\tau}, S_3(\tau) = \tau - 1, S_4(\tau) = \frac{1}{1-\tau}, S_5(\tau) = \frac{\tau-1}{\tau}, S_6(\tau) = \frac{\tau}{1-\tau}$$

One can check that Δ is mapped to the six shaded regions by $S_k, 1 \leqslant k \leqslant 6$.

There also exist six other mutually incongruent transformations S_k' which map Δ' to the six unshaded regions

$$S_1'(\tau) = \tau, S_2'(\tau) = -1 + \frac{1}{\tau}, S_3(\tau) = \tau + 1, S_4'(\tau) = \frac{1}{\tau}, S_5'(\tau) = -\frac{1}{1+\tau}, S_6'(\tau) = \frac{\tau}{\tau+1}$$

These 12 shaded and unshade regions form $\overline{\Omega} \cup \overline{\Omega'}$.

Take $\tau \in \mathscr{H}$. By Theorem 2, $\exists\ S \in SL_2(\mathbb{Z}), S\tau \in \overline{\Delta} \cup \overline{\Delta}'$ □

Corollary 3.1. *$\mathscr{H} \xrightarrow{\lambda} \mathbb{C} \setminus \{0,1\}$ is a covering map,i.e.,$\forall z \in \mathbb{C} \setminus \{0,1\}$, $\exists$ an open neighborhood U_0 of z_0* s.t. $\lambda^{-1}(U) = \bigsqcup_{\phi \in \Gamma(2)} U_\phi$ *where $\lambda\big|_{U_\phi} : U_\phi \to U$ is a homeomorphism.*

Chapter 2

Global analytic functions

3 月 2 日 51 分 43 秒

1 Analytic Continuation

1.1 Germs and sheaves

- 整体解析函数一般记作 **f**.
- (f, Ω)，**f** 的代表元，称作 **f** 的分支.
- **f** 在 Ω 上可能有不同的分支.

Let Ω be a region in $\mathbb{C}$ and $f\colon \Omega \longrightarrow \mathbb{C}$ an analytic function. Call pair (f, Ω) a function element. A global analytic function is a collection of function elements which are related to each other in the following manner.

Definition 1.1. *We call that the two function elements (f_1, Ω_1) and (f_2, Ω_2) are direct analytic continuations of each other iff $f_1 \equiv f_2$ in $\Omega_1 \cap \Omega_2 \neq \varnothing$.*

Remark. *There need not exist any direct cnotinuation of (f_1, Ω_1) to Ω_2, but if there is one, it is uniquely determined.*

Definition 1.2. *We say that $(\tilde{f}, \tilde{\Omega})$ is an analytic continuation of (f, Ω) iff $\exists$ a chain of function elements $(f_1, \Omega_1) = (f, \Omega), (f_2, \Omega_2), \cdots, (f_n, \Omega_n) = (\tilde{f}, \tilde{\Omega})$ s.t. (f_k, Ω_1) and (f_{k+1}, Ω_{k+1}) are direct continuations of each other.*

Hence we obtain an equivlence relation on $\{$function element$\}$, an equiv class is called a global analytic function.

(f, Ω) :*representative of*

Example 1.1. *open $D \subset \mathbb{C}$. Denote*

$$\mathfrak{S} = \mathfrak{S}_D = \{\text{analytic germ}(f, \zeta) : \zeta \in D, f \,\text{analytic near}\, \zeta\}.$$

Definition 1.3. *A sheaf $\mathfrak{S}$ over X is a topological space with a map $\pi : \mathfrak{S} \to X$ onto X such that*

(i) *π is a local homeomorphism.*

(ii) *For each $\zeta \in D$ the stalk $\pi^{-1}(\zeta) =: \mathfrak{S}_\zeta$ has the structure of an abelian group.*

(iii) *The group operations are continuous.*

FACT:$\exists$ a topology onthe sheaf $\mathfrak{S}_D$ of germs of analytic functions such that it satisfies the conditions of definition:

A subset $V \subset \mathfrak{S}_D$ is called open iff $\forall s_0 \in V, \exists (f, \Omega)$ such that

(1) π

Remark. *All function elements (f, Ω) form a base for the topology of $\mathfrak{S}_D$.*

Only verify condition (i): Use notions s_0

Define

$$\Delta := \{\mathbf{f}\}$$

1.2 Sections and Riemann surfaces

$\mathfrak{S} \xrightarrow{\pi} D$: sheaf over a topology space D.

Definition 1.4. *$\forall$ open $U \subset D$. Call a continuous map $U \xrightarrow{\varphi} \mathfrak{S}$ a section over U iff*

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & \mathfrak{S} \\ & \searrow^{\mathrm{Id}_U} & \downarrow \pi \\ & & D \end{array}$$

Since $\pi \circ \varphi = \mathrm{Id}_U$, φ is injective and $\varphi^1 = \pi\big|_{\varphi(U)}$.
Every section is a homeomorphism onto its image

- Every $s_0 \in \mathfrak{S}$ lies in the image $\varphi(U_0)$ of some section $U_0 \xrightarrow{\varphi} \mathfrak{S}$.

  By condition (i), we take an open neighborhood Δ of s_0 such that $U_0 := \pi(\Delta) \subset D, \pi\big|_\Delta$: $\Delta \longrightarrow U_0$ homeomorphism.

  Defing $\varphi = (\pi\big|_\Delta) : U_0 \longrightarrow \mathfrak{S}$. we have done.

- $\forall\ U \subset D$,define $\omega : U \to \mathfrak{S}, \zeta \mapsto 0\zeta$ easy to show that 0_U is continuous.

  Then 0_U is a section over U, called the zero section over U.

$$\Gamma(\mathfrak{S})$$

Remark. *If U is connected and $\varphi, \psi \in \Gamma(U; \mathfrak{S})$, then either $\varphi \equiv \psi$ on U or $\varphi(U) \cap \psi(U) = \phi$*

证明. Only need to show

- $\{\zeta \in U \mid \varphi(\zeta) = \psi(\zeta)\}$ open

  Assume that $s_0 = \varphi(\zeta_0) = \psi(\zeta_0)$ for some $\zeta_0 \in U$. By the definition of section,

  $$\varphi^{-1} = \pi\big|_{\varphi(U)}, \psi^{-1} = \pi\big|_{\psi(U)}$$

  Since $s_0 \in \Delta := \varphi(U) \cap \psi(U) \subset \mathfrak{S}$ open, $\varphi \equiv \psi$ over Δ.

- $\{\zeta \in U \mid \varphi(\zeta) \neq \psi(\zeta)\}$ open

  Assume that $\varphi(\zeta_0) = s_1 \neq s_2 = \psi(\zeta_0)$ for some $\zeta_0 \in U$. Since $\mathfrak{S}$ is Hausdorff, $\exists$ neighborhoods Δ_1, Δ_2 of $s_1, s_2 : \Delta_1 \cap \Delta_2 = \varnothing$

  Since $\varphi = (\pi\big|_{\Delta_1})$ over $\pi(\Delta_1) = U_1, \psi = (\pi\big|_{\Delta_2})$ over $U_2 := \pi(\Delta_2), \varphi(\zeta) \neq \psi(\zeta)$ for all $\zeta \in U_1 \cap U_2 \ni \zeta_0$.

□

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Example 1.2. *Sheaf of germs of continuous functions is non-Hausdorff.*

Let open $D \subset \mathbb{R}^n$. Using function elements $(f, \Omega), f \in C^0(\Omega), \Omega \subset D$, we can define germs of continuous functions over D and the corresponding sheaf $\mathfrak{S}$ which satisfieds the three conditions . But $\mathfrak{S}$ is non-Hausdorff. We give a particular counterexample for $\mathfrak{S}_{\mathbb{R}}$.

Let $f_1 \equiv 0$ and $f_2(x) = \begin{cases} 0 & x \leqslant 0 \\ x & x > 0 \end{cases}$ which gives germs 0 and $(f_2, 0)$ at the orign. Obviously, $0 \neq (f_2, 0)$ in $\mathfrak{S}$

Clearly 0 and $(f_2, 0)$ can't be separated by open sets!

We always consider the sheaf $\mathfrak{S}_D$ of analytic functions over $D \subset \mathbb{C}$.

Proposition 1.1. *A component of* $\mathfrak{S}$ *can be identified with a global analytic function.*

证明. Step 1

Let (f_1, Ω_1) be a direct continuation of (f_0, Ω_0) and Δ_0, Δ_1 be the sets of germs determined by $(f_0, \Omega_0), (f_1, \Omega_1)$

Since $\Delta_0 \simeq \Omega_0, \Delta_1 \simeq \Omega_1, \Omega_0 \cap \Omega_1 \neq \varnothing \Longrightarrow \Delta_1 \cap \Delta_2 \neq \varnothing, \Delta_1, \cup \Delta_2$ is connected.

Hence,all the function elements obtained from (f_0, Ω_0) by a chain of direct continuations determine germs contained in the component of $\mathfrak{S}_0$ of s_0.

Step2

Let $\mathfrak{S}_0'$ be the set of germs in $\mathfrak{S}_0$ determined by an analytic continuation

Since both $\mathfrak{S}_0'$ and its complement in $\mathfrak{S}_0$ are open in $\mathfrak{S}_0 \Longrightarrow \mathfrak{S}_0' = \mathfrak{S}_0$

Summing up the obove, we see that $\mathfrak{S}_0$ consists of exactly all the germs belonging to a global analytic function □

Definition 1.5. *Let* $\mathbf{f}$ *be the global analytic function obtained from* $s_0 \in \mathfrak{S}$

Call $\mathfrak{S}_0 =: \mathfrak{S}_0(\mho)$ *the Riemann surface of* $\mathbf{f}$.

There is a nature

$$\mathfrak{S}_0(\mho) \xrightarrow{\pi} D$$
$$\mho_\zeta \longmapsto \zeta$$

Look at Riemann surfaces as the natural world where analytic functions are alive, $\mathbf{f}$ can be looked at as a single-valued analytic function on $\mathfrak{S}_0(\mathbf{f})$, its value at $\mathbf{f}_\zeta$ being the constant term in the power series associated with $\mathbf{f}_\zeta$.

Given two global analytic functions $\mathbf{f}, \mathbf{g}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{S}_0(\mathbf{f}) & \xrightarrow{\quad\theta\quad} & \mathfrak{S}_0(\mathbf{g}) \\ & {\scriptstyle \pi_f} \searrow \quad \swarrow {\scriptstyle \pi_g} & \\ & D & \end{array}$$

then $g \circ \theta$ is a single-valued function on $\mathfrak{S}_0(\mathbf{f})$.

In the way, $\mathbf{f}', \mathbf{f}'', \cdots$ are all well defined on $\mathfrak{S}_0(\mathbf{f})$.

Example 1.3. *All entire functions live on* $\mathfrak{S}_0(\mathbf{f}), \forall\ \mathbf{f}$.

Example 1.4. *If* $\mathbf{g}, \mathbf{h}$ *are defined on* $\mathfrak{S}_0(\mathbf{f})$*, so are* $\mathbf{g} + \mathbf{h}$ *and* $\mathbf{gh}$.

Permanence principle

Suppose that $(f, \Omega), (g, \Omega), (h, \Omega), \cdots$ could be continued whenever (f, Ω) can be through a chain of direct continuation.

Assume that $G(f, g, h, \cdots) = 0$ on Ω. Then $G(\mathbf{f}, \mathbf{g}, \mathbf{h}, \cdots) = 0$ i.e. the same relation holds for all analytic continuations.

In particular, if a germ satisfies a polynomial differential equation, $G(z, f, f', f'', \cdots .f^{(n)}) = 0$, then the global analytic function **f** satisfies the same equation.

1.3 Analytic continuation along arcs

Given $[a,b] \xrightarrow{\gamma} \mathbb{C}$ arc in the complex plane, an arc $\bar{\gamma}:[a,b] \to \mathfrak{S}_0(\mathbf{f})$ is called an analytic continuation of $\mathbf{f}$ along γ iff $\pi \circ \bar{\gamma} = \gamma$.

Theorem 1.1. *Two lifting $\bar{\gamma}_1, \bar{\gamma}_2$ along γ are either identical or $\bar{\gamma}_1(t) \neq \bar{\gamma}_2(t)$ for all $a \leqslant t \leqslant b$.*

Remark. *A continuation along γ is uniquely determined by its initial value, germ $\bar{\gamma}(a)$ of form $\mathbf{f}_{\gamma(a)}$. Note that $\mathbf{f}$ may have several germs of this form.*

Can speak of the analytic continuation from an initial germ, provided the continuation exists.

singular path,singular point

It may happen that $\mathbf{f}$ doesn't have a continuation along γ, or that a continuation exists for some germs, but not for all. Consider an initial germ $\mathbf{f}_{\gamma(a)}$ which can't continue along γ. Define

$$\tau := \sup \left\{ t_0 > a : \exists \text{ continuation along} [a, t_0] \xrightarrow{\gamma} \mathbb{C} \right\}$$

Then $a < \tau \leqslant b$, and the continuation is possible for $t < \tau$, impossible for $t \geqslant \tau_0$. The subarc $\gamma([a,\tau])$ leads to a point where $\mathbf{f}$ ceases to exist.

We call this subarc a singular parts from the initial germ and it leads to a singular point over $\gamma(\tau)$.

Continuation along arcs v.s chains of direct continuations(stepwise continuations)

Roughly speaking, they are equivalent

- stepwise continuation$\Longrightarrow$ the one along an arc
- conversely, if γ and its lifting $\bar{\gamma}$ is given, we can find a chain of direct analytic continuations which produces the arc γ in the same way of LHS

Example 1.5 (Logarithmic log function)**.** *The set of all function element (f,Ω) with $\mathrm{e}^{f(\zeta)} = \zeta$ in Ω is global analytic function over $\mathbb{C}^{\times}$, denoted by $\log z$.*

证明. Only need to show that any two such function elements $(f_1,\Omega_1),(f_2,\Omega_2)$ can be joined by a □

Example 1.6. $\exists \gamma \subset \mathbb{C}$,$\exists$ *a global analytic function* $\mathbf{f}$

1.4 Homotopy curves

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1.5 The monodromy theorem

Consider a global analytic function $\mathbf{f}$ over $\Omega \subset \mathbb{C}$ such that for each arc $\gamma\colon [a,b] \to \mathbb{C}$ and each germ $(f_0, \zeta_0 = \gamma(a))$ of $\mathbf{f}$, there exists a continuation $\bar{\gamma}$ over γ.

Theorem 1.2. *Let $\gamma_1, \gamma_2\colon [a,b] \to \Omega$ be homotopic in Ω and have common endpoints. Suppose that a given germ of $\mathbf{f}$ at the initial point $\gamma_1(a) = \gamma_2(a)$ can be continued along all arcs in Ω. Then the continuations of the germ along γ_1 and γ_2 lead to the same germ at the terminal point.*

证明. Before the proof, we make the following observations.

(1) The continuation along $\gamma\gamma^{-1}$ leads back to the initial germ. Hence, the continuaiton along $\sigma_1(\gamma\gamma^{-1})\sigma_2$ has the smae effect as the one along $\sigma_1\sigma_2$.

That the continuations along γ_1 and γ_2 lead to the same

□

Corollary 1.1. *If Ω is simply connected, thne continuations of an initial germ (f, ζ_0) at $\zeta_0 \in \Omega$ of $\mathbf{f}$ can define a single-valued analytic function.*

1.6 Branch points

$\mathbb{D}_\rho^\times := \{z \in \mathbb{C} \mid 0 < |z| < \rho\}, \rho \in (0, +\infty]$

Fix $0 < z_0 = r < \rho$, Then the fundamental group of $\mathbb{D}_\rho^\times$ at the base z_0

$$\pi_1(\mathbb{D}_\rho^\times, z_0) = \left\{\text{homotopy class of the curves through} z_0 \text{in} \mathbb{D}_\rho^\times\right\} = \langle C \rangle \cong \mathbb{Z}.$$

Recall $\displaystyle\int_{C^m} \frac{\mathrm{d}z}{z} = 2\pi m\mathrm{i}, m \in \mathbb{Z}$

Assumption 1 Consider a global analytic function **f** that can be continued along each arc in $\mathbb{D}_\rho^\times$ e.g. $\sqrt{z}, \log z, \sqrt{z} + \log z$

Assume that **f** is not single-valued ,i.e., **f** has more than one germ at $z_0 = r$.

Choose an initial germ at $z_0 \in r$ and continue it along curves $C^m (m \in \mathbb{Z}^\times)$.

Then, either the continuation never comes back to the initial germ, or there exists a smallest positive integer h such that C^h leads back to the initial germ.

e.g. $h = 2, \infty$ for $\sqrt{z}$,$\log z$ resp.

Assumption 2 $\exists$ a smallest positive integer h greater than 1 such that C^h leads back to the initial germ.

Then if C^m also leads back to the initial germ, writting $m = nh + q (n \in \mathbb{Z}, 0 \leqslant q < h)$, we see that so does C^q, $q = 0$ and $h \mid m$.

Observation:Using the map $\mathbb{D}^\times$

2 Algebraic functions

3 月 16 日，3 月 21 日

Let $P \in \mathbb{C}[w, z]$. We interpret, for each z, the finite number of solutions

$$w_1(z), \cdots, w_n(z)$$

of $P(w, z) = 0$, as values of a global analytic function $\mathbf{f}(z)$, which is called an algebraic function.

Conversely, we shall tell whether a given global analytic function satisfies a polynomial equaiton.

2.1 The resultant of two polynomials in $\mathbb{C}[w, z]$

Definition 2.1. *Irreducible polynomial in* $\mathbb{C}[w, z]$.

Theorem 2.1. *Let $P(w, z)$ and $Q(w, z)$ be relatively primes in $\mathbb{C}[z, w]$ and have positive degree in w. Then*

$$\#\{z_0 \in \mathbb{C} \mid \exists\ w \in \text{ s.t. } P(w, z_0) = Q(w, z_0) = 0\} < \infty.$$

证明. We express P and Q according to decreasing powers of w. Assume $\deg_w P \geqslant \deg_w Q$

Using the Euclidean division algorithm, we have

$$c_0 P = q_0 Q + R_1$$

$$c_1 Q = q_1 R_1 + R_2$$

$$c_2 R_1 = q_2 R_2 + R_3$$

$$c_{n-1} R_{n-2} = q_{n-1} R_{n-1} + R_n$$

where $q_k, R_k \in \mathbb{C}[w, z], R_n \in \mathbb{C}[z]$

and $c_k \in \mathbb{C}[z]$ which are uesd to clean fractions.

CLAIM:$R_n(z) \not\equiv 0$

Assume that $P(w_0, z_0) = (w_0, z_0) = 0$. Then by (4). $R_n(z_0) = 0$ □

Definition 2.2. *Letting that the exponents c_k in (4) have the lowest degree possible, we could determing $R_n(z)$ uniquely, which is called the resultant of P and Q. Moreover, $R_n(z) = pP + qQ$ for some $p, q \in \mathbb{C}[z, w]$.*

Let $P \in \mathbb{C}[z, w]$ be irreducible. Then $P(w, z)$ and $P_w(w, z) = \dfrac{\partial P}{\partial w}(w, z)$ are relatively prime.

We call the resultent of P and P_w teh discriminant of P. The zeros of the discriminent are exactly

Corollary 2.1. *The above set coincides with* $\{z_0 \in \mathbb{C}, R_n(z_0) = 0\}$

the values z_0 for which the equation $P(w, z_0) = 0$ has multiple roots.

2.2 Definition and properties of algebraic functions

Definition 2.3. *A global analytic function* $\mathbf{f}$ *is called an algebraic function if and only if* $\exists\ P \in \mathbb{C}[z,w]$ *with* $\deg_w P > 0$ *such that all function elements* (f,Ω) *of* $\mathbf{f}$ *satisfy* $P(f(z),z) = 0$ *in* Ω.

Remark. *By the permanence principle, we could only assume the above equation for one function elements of* $\mathbf{f}$.

We may assume that $P(w,z)$ *is irreducible.*

Assume that $P(w,z)$ *is irreducible from now on.*

P *is uniquely determined by* $\mathbf{f}$ *up to a const.*

If $P \in \mathbb{C}[w]$, *by irreducibility, it must be of form* $w - a$, $\mathbf{f}$ *must be const.*

AIM Shall prove that $\exists$ an algebraic function corresponding to an irreducible polynomial $P(w,z)$ with $\deg_p P > 0$.

Let P have from $P(w,z) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z)$.

Set $C = \{z_0 \in \mathbb{C} \colon a_0(z_0) = 0$ or the discriminant $D(p)$ of P vanishes at$z_0\}$.

Then C is finite, say $C = \{c_1, c_2, \cdots, c_m\}$.

Fix $z_0 \notin C$, the equation $P(w,z_0) = 0$ has exactly n distinct roots, say $w_1, \cdots, w_n$.

Lemma 2.1. *Fix* $z_0 \notin C$, *There exists an open disk* Δ *centered at* z_0, *and* n *function elements* $(f_1,\Delta), \cdots, (f_n,\Delta)$ *such that*

(1) $P(f_j(z),z) = 0$

(2) $w_j = f_j(z_0)$

(3) *If* $P(w,z) = 0$ *for some function elememnt* $(w = w(z),\Delta)$, *then* $\exists\ 1 \leqslant j_0 \leqslant n$ *such that* $w(z) = f_{j_0}(z)$ *for some* j_0.

证明. Choose $0 < \varepsilon << 1$, such that disks $|w - w_j| < \varepsilon$ don't overlap. Denote by C_j the circles $|w - w_j| = \varepsilon$ where $P(w,z_0) \neq 0$.

By the argument principle,

$$\frac{1}{2\pi \mathrm{i}} \int_{C_j} \frac{P_w(w,z_0)}{P(w,z_0)} \mathrm{d}w = 1.$$

Moreover, the integrals define continuous functions near z_0, which can only take integer values. Hence, $\exists$ disk $\Delta \ni z_0$

$$\frac{1}{2\pi \mathrm{i}} \int_{C_j} \frac{P_w(w,z)}{P(w,z)} \mathrm{d}w = 1, \forall z \in \Delta$$

Hence the equation $P(w,z) = 0$ has exactly one root in $|w - w_j| < \varepsilon$,denoted by $f_j(z)$.

Moreover, by the residue theroem,

$$f_j(z) = \frac{1}{2\pi \mathrm{i}} \int_{C_j} w \frac{P_w(w,z)}{P(w,z)} \mathrm{d}w$$

which shows that $f_j(z)$ is analytic in Δ and $f_j(z_0) = w_j$. We have already proved (a) and (b). $\square$

Remark.

(1) *Each function element* (f, Ω) *satisfying* $P(f(z), z) = 0$ *in* Ω *is the direct continuation of* (f_j, Δ) *for some* $1 \leqslant j \leqslant n$.

(2) *A function element* (f, Ω) *satisfying* $P(f(z), z)$ *can be continued along all path in* $\mathbb{C} \backslash C$.

In order to show that the global analytic function $\mathbf{f}$ corresoponding to P is unique, we only need to show that all elements (f_j, Δ) belong to the same global analytic functions.

2.3 Behavior at the critical points

Choose $\delta > 0$ such that the disks $|z - c_k| < \delta, 1 \leqslant k \leqslant m$ don't overlap.

$z_0 = c_k + \frac{\delta}{2}$

Continuing germ (f_j, z_0) along $\tilde{C}$ leads to another germ (f_l, z_0).

Since $\exists$ n choices, we obtain a smallest positive integer $1 \leqslant h \leqslant n$ such that continuation along $\tilde{C}^h$ leads back to the initial germ.

By section 1.6, we have

$$f_j(z) = \sum_{\nu=-\infty}^{+\infty} A_\nu (z - c_k)^{\nu/h} \tag{5}$$

We make the following discussion according to the following three cases:

(1) $c_k \in \mathbb{C}, a_0(c_k) \neq 0$.

(2) $c_k \in \mathbb{C}, a_0(c_k) = 0$.

(3) Behavior at ∞.

$a_0(c_0) \neq 0$

We claim that $f_j(z)$ remains bounded as $z \to c_k$, i.e. f_j has at most an ordinary algebraic sigularity at c_k.

Otherwise, we could choose points $z_{\tilde{m}} \to c_k$ with $f_j(z_{\tilde{m}}) \to \infty$.

Without loss of generality, we assume $f_j(z_{\tilde{m}}) \neq 0$, by the equation $P(f_j(z_{\tilde{m}}, z_{\tilde{m}})) = a_0(z_{\tilde{m}}) f_j(z_{\tilde{m}})^n + \cdots + a_n(z_{\tilde{m}}) = 0$, we obtain

$$a_0(z_{\tilde{m}}) + a_1(z_{\tilde{m}}) f_j(z_{\tilde{m}})^{-1} + \cdots + a_n(z_{\tilde{m}}) f_j(z_{\tilde{m}})^{-n} = 0 \tag{6}$$

Letting $\tilde{m} \to \infty$, we find $a_0(c_k) = \lim_{\tilde{m} \to \infty} a_0(z_{\tilde{m}}) = 0$, Contradiction!

$a_0(c_k) = 0$

Take $l \in \mathbb{Z}_{>0}$ with $\lim_{z \to c_k} a_0(z)(z - c_k)^l \neq 0$

CLAIM $f_j(z)(z - c_k)^l$ remains bounded as $z \to c_k$ i.e. f_j has at most an algebraic pole at c_k.

Can prove by the similar contradiction argument.

Behavior at $z = \infty$

Recall $P(w, z) = a_0(z) w^n + a_1(z) w^{n-1} + \cdots + a_n(z), a_0, a_n \not\equiv 0$

We consider $\deg a_i = r_i$ and don't care about $a_i(z) \equiv 0$. Choose $l \in \mathbb{Z}_{>0}$ such that

$$l > \frac{1}{k}(r_k - r_0), \forall k = 1, \cdots, n \tag{2.1}$$

CLAIM As $z \to \infty, f_j z^{-l}$ remains bounded i.e. $f_j(z)$ has at most an algebraic pole at ∞.

Otherwise, we could choose $z_{\tilde{m}} \to \infty$ with $f_j(z_{\tilde{m}})^{-1} z_{\tilde{m}}^l \to 0$, which implies

$$f_j(z_{\tilde{m}})^{-k}$$

Multiplying (6) by $z_{\tilde{m}}^{-r_0}$, since $\deg a_j = r_j$, we find that

$$a_0(z_{\tilde{m}})z_{\tilde{m}}^{-r_0} \to 0$$

Since $r_0 = \deg a_0(z)$, $a_0(z) \not\equiv 0$, Contradiction!

Summing up, we have proved

FACT An algebraic function has at most finitely many algebraic singularity in $\overline{\mathbb{C}}$

We shall prove a converse of this fact.

Let $\mathbf{f}$ be global analytic function satisfying the following two conditions

(1) $\forall\ c \in \mathbb{C}$, $\exists$ a punctured disk Δ^* centered at c such that

- $\forall\ z_0 \in \Delta^*$, $\exists$ at least one and finitely many germs of $\mathbf{f}$ at z_0
- all germs of $\mathbf{f}$ at z_0 can be continued along all arcs in Δ^* and show algebraic character at c, i.e. $\exists$ the smallest positive integer h, $\exists\ \nu_0 \in \mathbb{Z}$, germs have form

$$\sum_{\nu=\nu_0}^{+\infty} A_\nu (z-c)^{\nu/h}$$

(2) For $c = \infty, \Delta^*$ is the exterior of a circle, $\exists h \in \mathbb{Z}_{>0}$ and $\nu_0 \in \mathbb{Z}$, each germ at $z_0 \in \Delta^*$ has form

$$\sum_{\nu=\nu_0}^{\infty} A_\nu z^{-\nu/h}$$

Remark. *Under the above conditions,* $\mathbf{f}$ *has finitely many effective singularity, which we denote by* $c_1, \cdots, c_n \in \overline{\mathbb{C}}$

Observation The number of germs at each point $z \in \{\}$

Denote by $f_1(z), \cdots, f_n(z)$ the branches of $\mathbf{f}$

3 Picard's theorem

3 月 23 日

Definition 3.1. *$a \in \mathbb{C}$ is called the lacunary value*（空隙值）*of a function $f(z)$ if $f(z) \neq a$ in its domain.*

Example 3.1. 0 *is the lacunary value of the entire function* e^z *on* $\mathbb{C}$*.*

Theorem 3.1 (Picard)**.** *A entire function with more than one finite lacunary value reduces to a const.*

证明. Let $\mathbb{C} \xrightarrow{f} \mathbb{C}$ be an entire function with at least two lacunary values $a \neq b$ in $\mathbb{C}$.

Without loss of generality, we assume $a = 0$ and $b = 1$.

Recall that modular function $\lambda : \mathscr{H} \to \mathbb{C}\backslash\{0,1\}$ is holomorphic and $\lambda'(\tau) \neq 0, \forall\ z \in \mathscr{H}$.

$$\begin{array}{ccc} & & \mathscr{H} \\ & & \downarrow \\ \mathbb{C} & & \mathbb{C}\backslash\{0,1\} \end{array}$$

Construct a global analytic function $\mathbf{h}$ whose function element (h, Ω) satisfy

(1) $\Im h(z) > 0$ and $\lambda(h(z)) = f(z), \forall z \in \Omega$

(2) $\mathbf{h}$ can be continued along all paths in $\mathbb{C}$

Since $\mathbb{C}$ simply connected, by the monodromy theorem, $\mathbf{h}$ defines an entire function taking values in $\mathbf{H}$ and is constant by Liouville's theorem. □

4 Linear differential equations

$$a_0(z)\frac{\mathrm{d}^n w}{\mathrm{d}z^n} + a_1(z)\frac{d^{n-1} w}{\mathrm{d}z^{n-1}} + \cdots + a_n(z)w = b(z) \tag{8}$$

$a_0(z), \cdots, a_n(z), b(z)$,entire functions, say polynomials.

We say that a global analytic function $\mathbf{f}$ solves (8) iff all function elements (f, Ω) of $\mathbf{f}$ satisfy the corresponding ODE in Ω.

Remark.

(1) *Recall in the real cese, we expect the equation*

$$a_0\mathbf{f}^{(n)} + a_1(z)\mathbf{f}^{(n-1)} + \cdots + a_n(z)\mathbf{f} = b \tag{9}$$

to have n linear independent solutions.

(2) *In the complex case, different local solutions can be function elements of the same global analytic function.*

In this case, the problem is to find out to what extent the local solutions are analytic continuations of each other.

The equation (8) is called homogeneous iff $b(z) \equiv 0$. Assume $a_k(z), 0 \leqslant k \leqslant n$, have no common zero.

Example 4.1. *In the case* $n = 1$

$$a_n \frac{\mathrm{d}w}{\mathrm{d}z} + a_1 w = 0 \Longleftrightarrow \mathrm{d}\log w = -\frac{a_1(z)}{a_0(z)}$$

$w = \exp\left(-\int \frac{a_1(z)}{a_0(z)}\mathrm{d}z\right)$

The problem is reduced to determining the multi-valued character of the integral $\int \frac{a_1(z)}{a_0(z)}\mathrm{d}z$ which is relevant to residue calculus.

We shall deal with second order linear differential equation, since they have all characteristic features of the general case.

4.1 Ordinary point

For differential equation

$$a_0(z)w'' + a_1(z)w' + a_2(z) = 0 \tag{10}$$

a point z_0 is called ordinary point iff $a_0(z_0) \neq 0$

$w'' = p(z)w' + q(z)w, p(z) = -\frac{a_1(z)}{a_0(z)}, q(z) = -\frac{a_2(z)}{a_0(z)}$ are analytic at z_0

Theorem 4.1. *If z_0 is an ordinary point of* (10)*, for any given $b_0, b_1 \in\ C$,$\exists!$ local solution (f, Ω) with $f(z_0) = b_0, f(z_1) = b_1$.*

In particular, the germ (f, z_0) is uniquely determined.

4.2 Regular singularity

A point z_0 with $a_0(z_0) = 0$ is called a singularity of of (10).

Then both $p(z)$ and $q(z)$ have poles at most at z_0.

The simplest case Assume that z_0 is a simple

Suppose that $w = b_0 + b_1 z + b_2 z^2$ is an analytic local solution of (11) near $z_0 = 0$. Then

4.3 Rieview

Consider

$$a_0(z)w'' + a_1(z)w' + a_2(z)w = 0, \tag{10}$$

$a_0 \not\equiv 0, a_1, a_2$:entire functions with no common zero

- ordinary point z_0: $a_0(z_0) \neq 0$ $\exists 2$ dim linear space of analytic solutions near z_0

Consider equation

$$w'' = pw' + qw \tag{11}$$

p, q meromorphic function in $\mathbb{C}$

Regular singularity z_0 of (11) p has at most a simple pole at z_0

q has at most a double pole at z_0

Recall the simplest case where both $p(z)$ and $q(z)$ have at most a simple pole at $z_0 = 0$.

We proved that if the Laurent development $\frac{p_{-1}}{z} + \cdots$ of $p(z)$ at $z_0 = 0$ satisfies $p_{-1} \notin \mathbb{Z}_{\geqslant 0}$ there exists a nontrivial analytic solution of (11) near $z_0 = 0$.

4.4 General regular singularity $z_0 = 0$

3 月 28 日

Suppose $w(z) = z^\alpha g(z)$ where g is analytic near $z_0 = 0$ and $g(0) \neq 0$.

Solves (11) for some $\alpha \in \mathbb{C}$ in some simply connected region Ω near $z_0 = 0$ but not containing z_0.

Then $g(z)$ satisfies

$$g'' = (p - \frac{2\alpha}{z})g' + (q + \frac{\alpha p}{z} - \frac{\alpha(\alpha-1)}{z^2})g \tag{18}$$

$p = \frac{p_{-1}}{z} + \cdots$

$q =$

Denote by α_1 and α_2 the roots of (19), called the (indicial) exponents of (11) at z_0

Then $\alpha_1 + \alpha_2 = p_{-1} + 1, \alpha_2 - \alpha_1 = p_{-1} - 2\alpha_1 + 1$

Hence α_1 is exceptional iff $\alpha - \alpha_1 \in \mathbb{Z}_{>0}$.

By symmetry, α_2 is exceptional iff $\alpha_2 - \alpha_1 \in \mathbb{Z}_{<0}$.

Hence, if the roots of the indical equation (19) don't differ by an integer,

5 Solutions at ∞

Suppose that $a_0 \not\equiv 0, a_1, a_2$ are polynomials without common zeros.

We investigate solution of

6 The hypergeometric differential equation

To study a 2nd DE with three regular singularities $0, 1, \infty$, we consider the equation

$$w'' = p(z)w' + q(z)w$$

with finite singularity at 0 and 1.

To make ∞ regular, $2z + z^2 p(z)$ must have at most a simple

7 Riemann's point of view

3 月 30 日

Riemann proved in 1857 that the solutions of hgde could be characterized by its nature.

Theorem 7.1. *The collection $\mathbb{F}$ of function elements (f, Ω) satisfying the following five characteristic features can be identified with the collection of local solutions of the hgDE*

$$w'' + \left(\frac{1-\alpha_1-\alpha_2}{z} + \frac{1-\beta_1-\beta_2}{z-1}\right)w' + \left(\frac{\alpha_1\alpha_2}{z^2} - \frac{\alpha_1\alpha_2+\beta_1\beta_2-\gamma_1\gamma_2}{z(z-1)} + \frac{\beta_1\beta_2}{(z-1)^2}\right)w = 0 \quad (23)$$

(1) $\mathbb{F}$ *is complete in the sense that it contains all continuations of* $(f, \Omega) \in \mathbb{F}$

(2) *The collection is linear.*

- $\forall (f_1, \Omega), (f_2, \Omega) \in \mathbb{F} \Longrightarrow (c_1 f_1 + c_2 f_2, \Omega) \in \mathbb{F}$
- *any three elements* $(f_1, \Omega), (f_2, \Omega), (f_3, \Omega) \in \mathbb{F}$ *linearly dependent.*

That is, $\mathbb{F}$ *has at most two dimension.*

(3) *The only finite singularity are at* 0 *and* 1, *and* ∞ *may be a singularity. Precisely, any element* $(f, \Omega) \in \mathbb{F}$ *can be continued along each path in* $\mathbb{C}\backslash\{0, 1\}$.

(4) $\exists$ *functions in* $\mathbb{F}$ *which behave like* z^{α_1} *and* z^{α_2} *near* 0, *like* $(z-1)^{\beta_1}$ *and* $(z-1)^{\beta_2}$ *near* 1 *and like* $z_{-\gamma_1}$ *and* $z^{-\gamma_2}$ *near* ∞.

(5) *Assume* $\alpha_2 - \alpha_1, \beta_2 - \beta_1, \gamma_2 - \gamma_1 \notin \mathbb{Z}$.

Remark.
- $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 1$ *can be deduced from the proof.*
- *The non-integral assumption* 5 *may be removable in some sense.*

Remark. *Riemann used the symbol* $P\left(\ ...\right)$

证明. We divide the proof of the theorem into four steps

(1) $\forall$ simply connected region $\Omega \subset \mathbb{C}\backslash\{0, 1\}$, $\exists$ two linearly independent elements $(f_1, \Omega), (f_2, \Omega) \in \mathbb{F}$

(2) Choose a third element $(f, \Omega) \in \mathbb{F}$. Then $\exists c, c_2, c_2 \in \mathbb{C}$ not all zero

$$\left\{ \ ... \quad \Longrightarrow \begin{bmatrix} f & f_1 & f_2 \\ f' & f_1' & f_2' \\ f'' & f_1'' & f_2'' \end{bmatrix} \equiv 0 \right.$$

We write the DE in form $f'' = p(z)f' + q(z)$ □

Part II

Kazaryan

Chapter 3

Preliminaries

1 $\mathbb{CP}^n$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+1} \setminus \{0\}$
$\forall\ A \in GL(n+1, \mathbb{C})$, A and λAhave the same effect on $\mathbb{P}^n$, where $\lambda \in \mathbb{C}^\times$

2 Coverings

Without loss of generality, we only consider connected surfaces.

Call a continuous map $M \xrightarrow{p} N$ a covering iff it satisfies the following three conditions

(1) every point y of N has a neighborhood $U = U_y \subset N$ whose p-preimage is a disjoint union of several copies of U

(2) the restriction of p to each copy is a homeomorphism

(3) either every point of N has countably many preimages, or the set of preimageo of every point is finite and any two points have the same number of preimages.

The common number of preimages is called the degree or the number of sheets of the covering.

Example 2.1.

(1) *$z \xrightarrow{p} z^n$ is an n-sheeted covering from $\mathbb{D}_\rho^\times = \{0 < |z| < \rho\}$ to $\mathbb{D}_{\rho^n}^\times$.*

Theorem 2.1. *Let M, N be compact surface and $M \xrightarrow{p}$ a n-sheeted covering. Then*

$$\chi(M) = n\chi(N).$$

证明. □

Monodromy of covering

Example 2.2. 内容...

3 Ramified coverings

Chapter 4

Algebraic curves

Complex algebraic curves = curves defined by homogeneous polynomial equations in complex projective space

1 Plane algebraic curves

$C = \left\{ (x, y, z) \in \mathbb{P}^2 \colon F(x, y, z) = \sum_{i+j+k=n} a_{ijk} x^i y^j z^k = 0, notalla_{ijk}vanishes \right\}$
$n \colon degreeofF$

In each of the three affine charts $x = 1, y = 1$ or $z = 1$, we could express the curve by a non-homogeneous equation in the remaining two variables.

Example 1.1. $\{(x, y, z) \in \mathbb{P}^2 \colon x^2 + y^2 - z^2 = 0\}$ *in chart* $z = 1$ *looks like* $\{(x, y) \in \mathbb{C}^2 \colon x^2 + y^2 = 1\}$.

If all coefficients a_{ijk} are real, the curve is called real.

The real point $(x : y : z)$ lying on a real curve form the real part of the curve, which may be empty, e.g. $x^2 + y^2 + z^2 = 0$.

Sometime, we use real parts of real curves to see a picture of the curve.

Example 1.2 (line)**.** $ax + by + cz = 0$, *where* $(a : b : c) \in \mathbb{P}^2$

For any pair $(x_1 : y_1 : z_1) \neq (x_2 : y_2 : z_2) \in \mathbb{P}^2, \exists!$ line through then given by $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$, rank$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 2$

Any two distinct lines intersect in exactly one points.

Any two different lines are given by $a_1x + b_1y + c_1z = 0, a_2x + b_2y + c_2z = 0$, where rank$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = 2$

Hence $\exists!$ solution $(x : y : z) \in \mathbb{P}^2$

1.1 Irreducible/reducible curve

4 月 11 日 20 分 57 秒

Call $C\colon F(x,y,z)=0$ irreducible iff $F\neq F_1F_2$ where F_1,F_2 have positive degrees.

Otherwise we call C reducible. In the latter case, as sets, the reducible curve $\{F_1F_2=0\}$ is the union of the curves $\{F_1=0\}$ and $\{F_2=0\}$. It may happen that $F_1=F_2$.

4 月 11 日 25 分 51 秒

Example 1.3 (Toy model of Bezout Theorem)**.** *Let $l_1,\cdots,l_n$ be pairwise distinct linear functions.*

Then the euqation $l_1l_2\cdots l_n=0$ gives the simplest reducible curves of degree n which is the union of the n lines $l_1=0,\cdots,l_n=0$.

Example 1.4. *Consider curves $l_1\cdots l_m=0$ and $l_1'\cdots l_n'=0$ such that $l_1=0,\cdots,l_n'$ are $(m+n)$ distinct lines in $\mathbb{P}^2$ and any three of these lines do not intersect at one point.*

Then the two curves $l_1\cdots l_m=0$ and $l_1'\cdots l_n'=0$ have exactly mn pairwise distinct intersection points.

1.2 Singular/Smooth point

4 月 11 日 41 分 45 秒

Definition 1.1. *Point $(x_0:y_0:z_0)$ on curve $F(x,y,z)=0$ is called singular/smooth iff $\mathrm{d}F=\dfrac{\partial F}{\partial x}\mathrm{d}x+\dfrac{\partial F}{\partial y}\mathrm{d}y+\dfrac{\partial F}{\partial z}\mathrm{d}z$ vanishes/does not vainish at $(x_0:y_0:z_0)$.*

Remark. 隐函数定理确定全纯函数，可参考 *Donaldson*

Definition 1.2 (Nondegenerate homogeneous polynomial)**.** *Call a homogeneous polynomial $F(x,y,z)$ nondegenerate iff curve $F(x,y,z)=0$ contains no singular point. In this case, we call the curve $F(x,y,z)=0$ smooth.*

Remark. 成为一维复流形，紧黎曼面.

Example 1.5. *A reducible curve $F_1F_2=0$ cannot be smooth since each point lying in $\{F_1=F_2=0\}$ is singular on the curve.* 之后会证明 $\{F_1=F_2=0\}$ 不是空集，此处先承认.

Example 1.6. *$\exists$ irreducible nonsmooth curve $x^2z+y^3=0$, $(0:0:1)$ is singular on the curve.*

1.3 在仿射坐标卡中计算奇点

4 月 11 日 52 分 16 秒

How to check that a curve is smooth in some chart, say $z=1$?

Let $A\in C$ lie in the chart $z=1$ and $f(x,y)=F(x,y,1)$. Then the differential $\mathrm{d}f=\dfrac{\partial f}{\partial x}\mathrm{d}x+\dfrac{\partial f}{\partial y}\mathrm{d}y$ vanishes at A iff $\mathrm{d}F=0$ at A.

Actually, by the Euler identity, $x\dfrac{\partial F}{\partial x}+y\dfrac{\partial F}{\partial y}+z\dfrac{\partial F}{\partial z}=nF$.

If $\dfrac{\partial F}{\partial x}=\dfrac{\partial F}{\partial Fy}=0$ at $A\in\{z=1\}$, then we also have $\dfrac{\partial F}{\partial z}=0$ at A.

Remark. *An irreducible curve $C = \{F(x, y, z) = 0\}$ has at most finitely many singular points.*

但还不知道怎么证.

Example 1.7. (1) *Each point on $C = \left\{l^2 = (ax + by + cz)^2 = 0\right\}$ is singular.*

(2) *$F(x, y, z) = x^n + y^n + z^n$ is nondegenerate.*

(3) *$(0 : 0 : 1)$ is the unique singular point of $x^2 + y^2 = 0$.*

Example 1.8. *Let C be a smooth conic in $\mathbb{P}^2$. Then in an appropriate coordinate system, C has form $x^2 + y^2 + z^2 = 0$.*

Definition 1.3 (Ordinary double point)**.** *Let $(0, 0)$ be a singular point point of an affine curve given by a nonhomogeneous polynomial equation $f(x, y) = 0$. The Taylor development of f about $(0, 0)$ has form $f(x, y) = 0 + 0 + (ax^2 + 2bxy + c^2y^2) + \cdots$ we call $(0, 0)$ an ordinary double point iff the quadratic part of f is nondegenerate, i.e.*

Definition 1.4. *Let A be a singular point of curve $F = 0$, say $A = (0 : 0 : 1)$. Then in chart $\{z = 1\}$*

Example 1.9. (1) *Let $l_1, \cdots, l_k$ be linear functions vanishing at A. Then $mul_A(l_1 \cdots l_k = 0) = 0$*

1. *For both two curves $y^2 = x^2(x - 1)$ and $y^2 = x^2$, the $multi(0, 0) = 2$.*

1.4 title

How many points in $\mathbb{P}^2$ are required to uniquely determine a curve of deg n pathing through them?

The space of curves of degree n in $\mathbb{P}^2 = \mathbb{P}^d$

2 第八周周三

Recall

Geometric question: How many points in $\mathbb{P}^2$ are required to uniquely determine a curve of degree n through them?

The Veronese embedding of $\mathbb{P}^2$ is defined to be

$$v_n \colon \mathbb{P}^2 \to \mathbb{P}^d = \mathbb{P}^{n(n+3)/2}, (x : y : z) \mapsto (\cdots, x^i y^j z^k, \cdots), \text{where } i + j + k = n, i, j, k \in \mathbb{Z}_{\geqslant 0}$$

The image under v_n of a curve of degree n in $\mathbb{P}^2$ is the cross-section of $v_n(\mathbb{P}^2)$ by a hyperplane. $v_n \colon \mathbb{P}^2 \to \mathbb{P}^d$ is nonedgenerate, i.e. $v_n(\mathbb{P}^2)$ is not contained in any hyperplane H in $\mathbb{P}^d = 0$.

Otherwise, there exists a curve of degree n which coincides with $\mathbb{P}^2$, Contradict with the Nullstellensatz.

Answer to the geometric question by the following 2 observations

Observation 1. There exist a curve of degree n through any given $\dfrac{n(n+3)}{2}$ points in $\mathbb{P}^2$

Moreover, the curve is unique iff $\operatorname{rank}(v_n(P_1), \cdots, v_n(P_d)) = d$.

Observation 2. There exists $(d + 1)$ points in $\mathbb{P}^2$ such that there exists no curve of degree d through them.

Observation 3. Suppose $P_1, \cdots, P_{d-1} \in \mathbb{P}^2$ satisfy $\operatorname{rank}(v_n(P_1), \cdots, v_n(P_d)) = (d - 1)$

Then there exist two distinct curves $F = 0$ and $G = 0$ of degree n such that each degree d curve through $P_1, \cdots, P_{d-1}$ has form $\lambda F + \mu G = 0$

We call the family $\lambda F + \mu G = 0$a pencil of curves of degree n.

3 Bezout's theorem and its applications

Common point A of two curves $F = 0$ and $G = 0$

(1) Assume that A is a smooth point for both two curves

  (a) transversal intersection

  (b) touch

(2) A is a singular point for one of them

  (c)

  (d)

Stability
Transversality is stable under small perturbation.
Tangency is an unstable configuration.
Enumeration of the intersection points of curves

Example 3.1. *Consider a line l and a curve $F(x, y, z) = 0$ of degree n. Choose two distinct points $(x_0 : y_0 : z_0), (x_1 : y_1 : z_1)$ in l. Then we can prarmetrize the line by the map*

$$\mathbb{C} \cup \infty = \mathbb{P}^1 \xrightarrow{\varphi} \mathbb{P}^2, t \longmapsto (x_0 + x_1 t : y_0 + y_1 t : z_0 + z_1 t)$$

Then we obtain the equation of the intersection points of the line and the curve $F(x, y, z) = 0$

$$F(x_0 + x_1 t, y_0 + y_1 t, z_0 + z_1 t) = 0$$

deg n in t.

The image under φ of the n roots of $(*)$ are the n intersection points. Counected with multiplicities.

The intersection of a curve $F(x, y, z) = 0$ of degree 3 and a curve

$$F(x, y, z) = a_0 y^3 + a_1(x, z)y^2 + a_2(x, z)y + a_3(x, z)$$

$$G(x, y, z) = b_0 y^2 + b_1(x, z)y + b_2(x, z)$$

a_k, b_k are homogeneous polynomial of degree k in x and z.

Without loss of generality, $a_0 b_0 = 0$,i.e. non of the two curves passes throught $(0 : 1 : 0)$.

Trivial observation: A point $(x_0 : y_0 : z_0)$ is an intersection point of $F = 0$ and $G = 0$ iff the two polynomials $F(x_0, y, z_0)$ and $G(x_0, y, z_0)$ have a common root y_0.

Crucial observation Let $\varphi = \varphi(y), \psi = \psi(y) \in \mathbb{C}[y] \setminus \{0\}$ be of degree m and n, resp. Then they have a common root iff $\exists \varphi_1, \psi_1 \in \mathbb{C}[y] \setminus \{0\} : \varphi\psi_1 = \psi\varphi_1$ and *deg*

证明. □

Let f_1 and g_1 have form $f_1(y) = u_0 y^2 + u_1 y_+ u_2, g_1(y) = v_0 y_v)1$ and satisfy

$$F(x_0, y_0, z_0)g_1(y) = G(x_0, y, z_0)f_1(y)$$

4 4 月 18 日第九周周一

4.1 Topological proof of Bezout's theorem

Step 1

Trivial observation: An integer valued continuous function on a connected space X is constant.

Consider in $\mathbb{P}^2$

C_1: m distinct lines through one point

C_2: n distinct lines through another point

$C_1 \cap C_2 = \{m \cdot n\}$

Slightly perturbing their coefficients of C_1 and C_2, we obtain a pair of curves with mn transversal intersection points.

Step 2

In the space $\mathbb{P}^{\frac{m(m+3)}{2}} \times \mathbb{P}^{\frac{n(n+3)}{2}}$ of pairs of curves of deg m and n resp, there is a Zariski open subset of pairs of curves with $m \cdot n$ transversal intersection points.

5 Rational parametrization

Observation: A conic is smooth iff it is irreducible.

Rational parametrization of a smooth conic C

Take a point $A \in C \subset \mathbb{P}^2$.

All line through A in $\mathbb{P}^2$ form a projective line.

Coordinate representation of this parametrization

The conic $x^2 + y^2 - z^2 = 0$ is given by $x^2 + y^2 = 1$ in affine chart $\{z = 1\}$.

Line $y = t(x+1)$ through A intersectes the conic at another point $\left(\dfrac{1-t^2}{1+t^2}, \dfrac{2t}{1+t^2}\right)$.

The corresponding homogeneous version has form $(s:t) \mapsto (s^2 - t^2 : 2st : s^2 + t^2)$

rational normal curve of deg 2.

$\mathbb{P}^1 \to \mathbb{P}^2$

Example 5.1.

-
- *An irreducible cubic has at most one singularity point of multiplicity 2.*

Theorem 5.1. *An irreducible curve of deg n has at most $N = \dfrac{(n-1)(n-2)}{2}$ ordinary double points.*

证明. 内容... □

Remark. $\exists$ *an irreducible curve of deg n with exactly $N = \dfrac{(n-1)(n-2)}{2}$ double points.*

We shall show that a deg n irreducible curve with N double points admits a rational parametrization,i.e., $\exists$ homogeneous polynomials $x(s,t), y(s,t), z(s,t)$ of deg n such that the image of the mapping

$$\mathbb{P}^1 \to \mathbb{P}^2, (s:t) \mapsto (x(s,t) : y(s,t) : z(s,t))$$

coincides with C.Moreover, different values of $(s:t)$ yield different points in C except each double point of C has exactly two preimages.

($\mathbb{P}^1 \to C$ is called a normalization of C)

Topologically, $C \setminus \{\text{double points}\} \cong \mathbb{S}^2 \setminus \{2N\text{points}\}$

Theorem 5.2. *An irreducible degree $n \geqslant 3$ curve with N double points admits a rational parametrization.*

证明. Deal with case $n = 3$ at first.

Via some projective transformation, we can assume that the irreducible cubic has equation $y^2 = x^2 + x^3$ in affine chart $\{z = 1\}$ with the unique double point $(0,0) =: O$.

Line $y = tx$ through O meets the cubic in exactly one other point $(x(t), y(t)) = t^2 - 1, t(t^2 - 1)$

Then $\mathbb{P}^1 \to C, t \mapsto (t^2 - 1, t(t^2 - 1))$

- to every point of C except O there corresponds exactly one value of t.

- to O there correspond two tangents $y = \pm x$ to the two branches of C at O, i.e. the exceptional point of C is the double point O.

Given a partial proof for case $n > 3$. Let $A_1, \cdots, A_N$ be double points of C. choose arbitrary points $P_1, \cdots, P_{n-3}$ on C different from P_j. Since $N + (n-3) = \dfrac{(n-2)(n+1)}{2} - 1$, there exists a pencil of deg $n-2$ curves through $A_1, \cdots, A_N, P_1, \cdots, P_{n-2}$.

Each curve of deg $n-2$ from the pencil has $2N + (n-3)$ exactly one other common point. On the other hand, the point $A_1, \cdots, A_N, P_1, \cdots, P_{n-3}$ and every onter point on C determines a unique curve of deg $(n-2)$ from the pencil. Thus we obtain a parametrization of $C \setminus \{A_1, \cdots, A_N, P_1, \cdots, P_{n-3}\}$ by the parameter of the pencil.

HW: The left part of the proof. □

5.1 Nontransversally intersecting pairs of plane curves

Supplement ot [P23.KLP] where the authors give a topological proof for Bezout theorem.

Let $m, n \in \mathbb{Z}_{>0}$. Recall that the plane curves of degree m form a projective space of dim $\dfrac{m(m+3)}{2}$.

Easy observation. Pairs of non-transversally intersecting lines form the diagnal of $\mathbb{P}^2 \times \mathbb{P}^2$.

Let $\max(m, n) > 1$.

Claim:Non-transversally intersecting curves of deg m and deg n in $\mathbb{P}^2$ form a hypersurface in $\mathbb{P}^{\frac{m(m+3)}{2}} \times \mathbb{P}^{\frac{n(n+3)}{2}}$.

Proof of sketch. We deal with case $(m, n) = (3, 2)$.

Suffice to show the statement locally.

Chapter 5

Complex structures and the topology of curves

4 月 18 日 1 小时 23 分 5 秒

Implicit function theorem

4 月 18 日 1 小时 32 分 43 秒

1 The complex structure on a curve

4 月 18 日 1 小时 45 分 19 秒

Remark. 此书之前没有定义过 $\mathbb{P}^n$ 中的光滑曲线. 可先按 $n = 2$ 理解.

Assume that open $W \subset \mathbb{C}^{n+1} \times \mathbb{C}, n \geqslant 2$ and $W \xrightarrow{f} \mathbb{C}^{n+1}$ holomorphic function.

Let $(w^0; z_0)$

Hence, by the implicit function theorem, we can identify a neighborhood of each point of C with $\mathbb{D} \subset \mathbb{C}$ i.e. $\forall\, A \in C, \exists$ a one-to-one map m_U from a neighborhood a neighborhood $U = U_A \subset \mathbb{C}$ onto $\mathbb{D}$ which is called a local coordinate in U. If two such neighborhooods have a nonepmty intersection, then the mapping $m_U \cdot m_V^{-1}$, defined in a subdomain of $\mathbb{D}$ is biholomorphic.

4 月 18 日第二段 1 分 26 秒

Example 1.1.

4 月 18 日第二段 4 分 58 秒

Definition 1.1. 全纯映射

4 月 20 日 29 分 40 秒

Example 1.2. 内容...

2 The genus of a smooth plane curve

4 月 20 日 34 分 0 秒

思路：给定一条次数 n 的光滑曲线，造一个合适的到 $\mathbb{P}^1$ 的分歧覆盖，数退化的信息，利用 Riemann-Hurwitz 公式，得到亏格与次数之间的关系.

Remark. *一般曲线的亏格也能算，可参考 Kirwan 的 7.3 节.*

2.1 证明二

4 月 20 日 1 小时 19 分 42 秒

2.2 定理 2.6 新证

4 月 20 日 1 小时 24 分 20 秒

3 4 月 25 日第十周周一

Let C be smooth curve.

Definition 3.1. *Double tangent*

Definition 3.2 (flex/inflection)**.** *A point on C is a flex if mult$(l \cap C) \geqslant 3$, where l tangent at A to C.*

Theorem 3.1. *$\exists$ exactly $n^2 - n$ distinct tangents from a point $\in \mathbb{P}^2$ in general position to C.*

证明. Choose a point $P \in \mathbb{P}^2$ which lies neither on C, nor on double tangents, nor on tangents at inflection points.

Consider the ramified covering $p\colon C \to \mathbb{P}^1$, each of whose ramification points has exactly $(n-1)$ preimages. By R-H,

$$3n - n^2 = \chi(C) = 2n - \sum_{j=1}^{k}[n-(n-1)], k = \#\left\{\text{ramification points of } C \xrightarrow{p} \mathbb{P}^1\right\}$$

□

Corollary 3.1. *From a point in general position on C, there are exactly $n^2 - n - 1$ distinct tangents to C.*

Remark. *For a general point $A \in C$, there are $n^2 - n - 2$ tangents through A beside the one to C at A.*

As $n = 3$, from a general point A on a smooth cubic c, $\exists$ 4 tangents.

3.1 j-invariant of smooth cubics

Let C be a smooth cubic in $\mathbb{P}^2$.

Fact1: Through each point A of C, there are 4 pairwise distinct tangents to C which differ from the tangent at A except inflections.

Definition 3.3 (&**Fact2**)**.** *$\forall x \in C$, the quadruple of the four tangents through x to C determines 4 points in the pencil of lines through x, say $a, b, c, d \in \mathbb{P}^1 = \mathbb{C} \cup \infty$. Define their cross ratio to be*

$$[a, b, c, d] := \frac{c-a}{c-b} : \frac{d-a}{d-b} \in \mathbb{C} \setminus \{0, 1\}$$

which depends on the order of the four points, but not on the coordinates of these points on $\mathbb{P}^1$.

Remark. 任意两个坐标系之间都只差一个 *mobius* 变换吗？

Fact3 Denote $\lambda = [a, b, c, d]$. Then

$$J(\lambda) := \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1-\lambda)^2}$$

does not depend on the order of the four points.

Fact4 Denote by $\lambda(x)$ the corss ration of the four tangents from $x \in C$ to C. Then $J\colon C \to \mathbb{C} \cup \{\infty\}, x \mapsto J(\lambda(x))$ is holomorphic and constant, denoted by $J(C)$.

Remark. *Under a projective transformation* $\varphi \in \mathrm{PGL}(3, \mathbb{C})$ *of* $\mathbb{P}^2$*, tangents to* $x \in C$ *goes to tangents at* $\varphi(x)$ *to* $\varphi(C)$*.*

The quadruple of 4 tangents to x also undergoes a projection transformation from the pencil of lines through x to the one through $\varphi(x)$. Hence, $J(C) = J(\varphi(C))$, i.e. $J(C)$ is a projective holomorphic invariant of C.

Fact5 The J-invariant of cubic $y^2 = x(x-1)(x-\lambda), \lambda \in \mathbb{C} \setminus \{0, 1\}$

In particular, if $J(\lambda_1 \neq J(\lambda_2))(\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0, 1\})$ then the two cubics $y^2 = x(x-1)(x-\lambda_1)$ and $y^2 = x(x-1)(x-\lambda_2)$ can't be projectively equivalent.

4 Hessian and inflection points

Let $F(x,y,z)$ be an irreducible homogenous polynomial of $\deg n \geqslant 2$ and A a smooth point on $C : F(x,y,z) = 0$.

The tangent line l at A to C has equation $x\dfrac{\partial F}{\partial x}(A) + y\dfrac{\partial F}{\partial y}(A) + z\dfrac{\partial F}{\partial z} = 0$.

l intersecs C with multiplicity $\geqslant 2$.

Definition 4.1. *A is called a flex of C iff this multiplicity is > 2.*

A is called an ordinary flex of C iff this multiplicity is $= 3$.

Definition 4.2. *The Hessian H_F of F to be*

$$H_F(x,y,z) = \det\begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{xz} & F_{yz} & F_{zz} \end{pmatrix}, \deg H_F = 3(n-2).$$

Exercise 3.7: A is a flex of C iff $H_F(A) = 0$.

We need a lemma to show it.

Lemma 4.1.

$$z^2 H_F(x,y,z) = (n-1)^2 \begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & \dfrac{nF}{n-1} \end{vmatrix}$$

证明. $nF = xF_x + yF_Y + zF_z$

$n - 1F_x = xF_{xx} + yF_{xy} + zF_{xz}$

Then

$$zH_F(x,y,z) = \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ zF_{xz} & zF_{yz} & zF_{zz} \end{vmatrix} \xlongequal{1\times x+2\times y\to 3} (n-1)\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{xy} & F_{yy} & F_{yz} \\ F_x & F_y & F_z \end{vmatrix} \xlongequal{1\times x+2\times y\to 3} \frac{(n-1)^2}{z}\begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & \dfrac{nF}{(n-1)} \end{vmatrix}$$

□

Solution to Exersics 3.7.

Assume $A \in \{z = 1\} \cap C$. Then $H_F(A) = 0 \overset{z_A=1}{\Longleftrightarrow} \begin{vmatrix} F_{xx} & F_{xy} & F_x \\ F_{xy} & F_{yy} & F_y \\ F_x & F_y & 0 \end{vmatrix} = 0$

$\Longleftrightarrow F_{xx}(F_x)^2 + F_{yy}(F_y)^2 - 2F_{xy}F_xF_y$

$\Longleftrightarrow A$ is a flex of $F(x,y,1) = 0$ in $\{z = 1\}$. □

Remark. *A smooth conic has no flex.*

A $\deg n$ irreducible curve has at most $3n(n-2)$ inflection points.

In particular, a smooth cubic has 9 inflection points.

5 Hyperelliptic curves

Definition 5.1. *We call a compact Riemann surface hyperelliptic if its geuns* > 1 *and it is a* 2*-sheeted ramified covering of* $\mathbb{P}^1$*.*

All ramification points are simple. By RH, $\#\{ramificationpoints\}$ *are even, say* $2k$*, then* $g(S) = k - 1$*.*

Example 5.1. *Let* $P_n \in \mathbb{C}[x]$ *be a polynomal of degree* n *without multiple roots with* $n \geqslant 3$*, say* $n = 2g + 1$ *or* $2g + 2$*.*

Consider the plane curve given by $y^2 = P_n(x)$ *in affine chart* $\{z = 1\}$*,* $d(y^2 - P_n(x))$ *nowhere vanishes in* $\{z = 1\}$*.*

In $\mathbb{P}^2$, C is given by $y^2 z^{n-2} = a_n x^n + a_{n-1} x^n z + \cdots + a_0 z^n$, and has point $(0 : 1 : 0)$ at infinity, which is smooth in C iff $n = 3$.

We shall modify C to a hyperelliptic Riemann surface. This process is called the Riemann compactification of $C \backslash (0 : 1 : 0) \subset \{z = 1\} = \mathbb{C}^2$