黎曼几何

孙天阳

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目录

Chapter 1

黎曼度量

1 定义和例子

2 Riemannian metric

$\gamma : (a, b) \to M$

$\int_a^b |\gamma'(t)| \mathrm{d}t = length(\gamma)$

Hilbert space$\Longrightarrow$ Riemannian geometry

Banach space$\Longrightarrow$ Finsler geometry

Just for the purpose of

$\gamma'(t), (v, x)$

$\|\gamma'(t)\|^2 = g_{ij} v^i v^j = (v^1, \cdots, v^d) \begin{pmatrix} g_{ij} \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix}$ bilinear form, (g_{ij}) positive definite,symmetric matrix

(U, y) $w^i \frac{\partial}{\partial y^i} = w^i \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$

$h_{ij}(y(p)) = g_{kl}(x(p)) \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j}$

(g_{ij}) $(0, 2)$ tensor! And we assume its coefficients are smooth on $x(U)$

定义 2.1. *A Riemannian metric g on a smooth manifold M is a smooth $(0, 2)$-tensor satisfying*

$$g(X, Y) = g(Y, X), \quad g(X, X) \geqslant 0 \& g_p(X, X) = 0 \Longleftrightarrow X(p) = 0$$

for any smooth tangent vector field X, Y.

A Riemannian manifold is a smooth manifold with a Riemannian metric.

例子 2.2. $\mathbb{R}^n$

- $(g_{ij}) = (\delta_{ij})$
- 球面几何 $(g_{ij}) = \frac{4}{(1 + \sum_{i=1}^n (x^i)^2)^2} (\delta_{ij})$
- 双曲几何 $(g_{ij}) = \frac{4}{(1 - \sum_{i=1}^n (x^i)^2)^2} (\delta_{ij})$

2.1 Existence of Riemannian metric

定理 2.3. *A smooth manifold has a Riemannian metric.*

Extrinsic proof. Whitney embedding

$f : M^n \to N^{n+k}$ smooth immersion ($\mathrm{d}f_p$ is injective)

Let (N, g_N) be a Riemannian metric

Pull-back metric f^*g_N on M

$$(f^*g_N)_p(X_p, Y_p) = g_N(\mathrm{d}f_p(X_p), \mathrm{d}f_p(Y_p))$$

□

Intrinsic proof. U_p coordinate neighborhood.$\{U_p, p \in M\}$ open cover.

paracompact$\Longrightarrow$ WLOG,let $\{U_\alpha\}$ be a locally finite covering of M by coordinate neighborhood.

Partition of unity $\{\varphi_\alpha\}$ subordinate to $\{U_\alpha\}$.

$x\colon U_\alpha \to x(U_\alpha) \subset \mathbb{R}^n$

$g_p(X, Y) = \sum_\alpha \varphi_\alpha(p)(g_\alpha)_p(X, Y)$. □

定义 2.4. *Let $(M, g_M), (N, g_N)$ be two Riemannian manifolds. $\varphi\colon M \to N$ is called an* ***isometry*** *if φ is a diffeomorphism and $\varphi^* g_N = g_M$.*

2.2 黎曼度量张量 $\rightsquigarrow$ 度量

定义 2.5. *A function $d\colon M \times M \to \mathbb{R}$ is called a metric if*

(i) $d(p,q) \geqslant 0$, *and* $d(p,q) = 0 \Longleftrightarrow p = q$.

(ii) $d(p,q) = d(q,p)$.

(iii) $d(p,q) \leqslant d(p,r) + d(r,q)$, $\forall\ r \in M$.

Let (M,g) be a Riemannian manifold, for any $p,q \in M$, consider
$C_{p,q} = \{\gamma : [a,b] \to M \mid \gamma$ piecewise smooth regular curve with $\gamma(a) = p, \gamma(b) = q\}$.
Define $d(p,q) = \inf\{Length(\gamma) \mid \gamma \in C_{p,q}\}$.
The following questions are immediate

(1) Is $C_{p,q}$ empty?

(2) Is $d(p,q) < +\infty$?

(3) Is d a metric?

(4) Can the infimum be attained?

Let $E_p = \{q \in M : p,q$ can be connected by a curve $\in C_{p,q}\}$. It is easy to show by connectedness argument that $E_p = M$. So $C_{p,q}$ could not be empty.

Take $\gamma \in C_{p,q}$, we can cover it by finite coordinate charts. So we just need to show any piecewise smooth curve contained in a coordinate chart has finite length.

$$Length(\gamma) = \int_a^b \sqrt{g_{ij}\frac{\partial x^i \circ \gamma}{\partial t}\frac{\partial x^j \circ \gamma}{\partial t}}\mathrm{d}t$$

引理 2.6.

Next we show $d(p,q)$ is a metric. It is obvious from definition that $d(p,q) \geqslant$ and $d(p,q) = d(q,p)$. Because we consider piesewise smooth curve, triangle inequality is also easy. If $p \neq q$, we can find a coordinate chart U of p such that $q \notin U$.

2.3 度量 $\rightsquigarrow$ 黎曼度量张量

https://mathoverflow.net/questions/45154/riemannian-metric-induced-by-a-metric

3 黎曼体积形式

4 商流形的黎曼度量

1

2

Chapter 2

寻找最短线

定义 0.1. 设 $c\colon [a,b]\to M$ 是一条光滑曲线. c 的一个（单参数）变分是指一个光滑映射

$$F\colon [a,b]\times(-\varepsilon,\varepsilon)\to M,\quad (t,s)\mapsto F(t,s)$$

满足 $F(t,0)=c(t)$. 记 $\dfrac{\partial F}{\partial t}=\mathrm{d}F\left(\dfrac{\partial}{\partial t}\right),\dfrac{\partial F}{\partial s}=\mathrm{d}F\left(\dfrac{\partial}{\partial s}\right)$（注意该记法与将 $\mathrm{d}c\left(\dfrac{\mathrm{d}}{\mathrm{d}t}\right)$ 记作 $c'(t)$ 的习惯相同）. 称沿 c 的向量场 $V(t):=\dfrac{\partial F}{\partial s}(t,0)$ 为变分场.

1 例子

Euclidean geometry

(r,θ)

$g=\mathrm{d}r\otimes\mathrm{d}r+r^2\mathrm{d}\theta\otimes\mathrm{d}\theta$

$\gamma:[a,b]\to M,\gamma(a)=p,\gamma(b)=q$

$Length(\gamma)=\int_a^b\sqrt{g(\gamma'(t),\gamma'(t))}\mathrm{d}t$

$(r(t),\theta(t)),r'(t)\dfrac{\partial}{\partial t}+\theta'(t)\dfrac{\partial}{\partial\theta}$

$$\begin{aligned}Length(\gamma)&=\int_a^b\sqrt{g(\gamma'(t),\gamma'(t))}\mathrm{d}t\\&=\int_a^b\sqrt{r'(t)^2+r(t)^2\theta'(t)^2}\mathrm{d}t\\&\geqslant\int_a^b|r'(t)|\mathrm{d}t\\&\geqslant\left|\int_a^b r'(t)\mathrm{d}t\right|\\&=|r(b)-r(a)|\end{aligned}$$

$=$ holds iff $\theta'(t)\equiv=0,\gamma(t)$ monotonic.

$S^2 \subset \mathbb{R}^3$

$$\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \in (0, 2\pi)$$
$$\left\{(\varphi, \theta) \mid \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2}), \theta \in (0, 2\pi)\right\}$$
$$g = \mathrm{d}\varphi \otimes \mathrm{d}\varphi + \cos^2\varphi \mathrm{d}\theta \otimes \mathrm{d}\theta$$

2 弧长泛函与能量泛函

设 (M, g) 是一个黎曼流形.

定义 2.1. 称光滑曲线 $\gamma\colon [a,b] \to M$ 是正则的如果 $\|\gamma'(t)\| \neq 0, \forall\, t \in I$.

分段光滑（正则）曲线

定义 2.2. *If $\gamma\colon I \to M$ is a smooth regular curve and if $p\colon I' \to I$ is a smooth map with non-zero derivative, then we say that $\gamma \circ p\colon I' \to M$ is a reparametrization of $\gamma\colon I \to M$.*

It is easy to check that any reparametrization of a smooth regular curve is still a smooth regular curve and this defines an equivalent relationship on the space of all smooth regular curves to M.

We will use **parametrized curve** to refer to a smooth regular curve and **curve without parametrization** to refer to an equivalent class of smooth regular curves under reparametrization.

Let $\gamma\colon [a,b] \to M$ be a parametrized curve, we can define its length

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \mathrm{d}t := \int_a^b \sqrt{g(\gamma'(t), \gamma'(t))} \mathrm{d}t.$$

It is easy to check that

引理 2.3. *If $\gamma \circ p\colon I' \to M$ is a reparametrization of $\gamma\colon I \to M$, then $L(\gamma \circ p) = L(\gamma)$.*

So we can actually define length for curves without parametrization.

弧长参数化

There is always a canonical representative element for any equivalent class of smooth regular curves under reparametrization.

命题 2.4. *Suppose $\gamma\colon I \to M$ is a parametrized curve.*

(1) $p\colon I \to [0, L(\gamma)], t \mapsto \int_a^t \|\gamma'(s)\| \mathrm{d}s$ *is a smooth map with non-zero derivative.*

(2) *Suppose $\gamma \sim \gamma'$, then $\gamma \circ p^{-1} = \gamma' \circ p'^{-1}$ as maps from $[0, L(\gamma)]$ to M.*

We call $\gamma \circ p^{-1}$ the ***arclength reparametrization*** *of γ.*

命题 2.5. *$\gamma\colon I \to M$ is parametrized with arclength iff $\|\gamma'(t)\| \equiv 1$.*

能量泛函

定义 2.6. 设 $\gamma\colon [a,b] \to M$ 是分段光滑正则曲线，定义

$$E(\gamma) = \frac{1}{2} \int_a^b \langle \gamma'(t), \gamma'(t) \rangle \, \mathrm{d}t$$

3 能量泛函的变分 I

设 (M,g) 是一个黎曼流形. 设 $\gamma\colon [a,b]\to M$ 是一条光滑曲线，F 是 γ 的一个变分.
任给 $y(t)$ 满足 $y(a)=y(b)=0$，

$$2E(\gamma_\varepsilon)=\int_a^b g_{ij}(x(t)+\varepsilon y(t))\frac{\mathrm{d}}{\mathrm{d}t}(x^i(t)+\varepsilon y^i(t))\frac{\mathrm{d}}{\mathrm{d}t}(x^j(t)+\varepsilon y^j(t))\mathrm{d}t$$

$$0=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}2E(\gamma_\varepsilon)=\int_a^b g_{ij,k}(x)y^k\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\mathrm{d}t+\int_a^b g_{ij}(x)\frac{\mathrm{d}y^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\mathrm{d}t+\int_a^b g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}y^j}{\mathrm{d}t}\mathrm{d}t$$

$$\int_a^b g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}y^j}{\mathrm{d}t}\mathrm{d}t=-\int_a^b\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)y^j\mathrm{d}t=-\int_a^b g_{ij,k}(x)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}y^j\mathrm{d}t-\int_a^b g_{ij}(x)\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}y^j\mathrm{d}t$$

$$\int_a^b g_{ij}(x)\frac{\mathrm{d}y^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\mathrm{d}t=-\int_a^b\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\right)y^i\mathrm{d}t=-\int_a^b g_{ij,k}(x)\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}y^i\mathrm{d}t-\int_a^b g_{ij}(x)\frac{\mathrm{d}^2x^j}{\mathrm{d}t^2}y^i\mathrm{d}t$$

$$0=\int_a^b\left(g_{ij,k}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-g_{ik,j}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}-g_{kj,i}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-2g_{ik}(x)\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}\right)y^k\mathrm{d}t$$

$$g_{ij,k}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-g_{ik,j}(x)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}-g_{kj,i}(x)\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}-2g_{lk}(x)\frac{\mathrm{d}^2x^l}{\mathrm{d}t^2}$$

$$2g_{lk}(x)\frac{\mathrm{d}^2x^l}{\mathrm{d}t^2}+(g_{ik,j}+g_{kj,i}-g_{ij,k})\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}=0$$

$$\frac{\mathrm{d}^2x^l}{\mathrm{d}t^2}+\frac{1}{2}g^{kl}(g_{ik,j}+g_{kj,i}-g_{ij,k})\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}=0$$

定义 3.1. 设 (M,g) 是黎曼流形，(U,x) 是一个坐标卡，g 在 (U,x) 下的分量表示为 (g_{ij})，定义 U 上的一族函数 $\Gamma_{ij}^k=\dfrac{1}{2}g^{kl}(g_{jl,i}+g_{il,j}-g_{ij,l})$，称作第二类 *Christoffel* 符号.

命题 3.2.

(1) $\Gamma_{ij}^k=\Gamma_{ji}^k$

(2) $g_{ij,l}=g_{kj}\Gamma_{il}^k+g_{ik}\Gamma_{jl}^k$

命题 3.3. $\widetilde{\Gamma}_{ij}^k=\Gamma_{\alpha\eta}^\gamma\dfrac{\partial x^\alpha}{\partial\tilde{x}^i}\dfrac{\partial x^\eta}{\partial\tilde{x}^j}\dfrac{\partial\tilde{x}^k}{\partial x^\gamma}+\dfrac{\partial\tilde{x}^k}{\partial x^\gamma}\dfrac{\partial^2x^\gamma}{\partial\tilde{x}^i\partial\tilde{x}^j}$

命题 3.4. $\dfrac{\mathrm{d}^2x^k}{\mathrm{d}t^2}+\Gamma_{ij}^k\dfrac{\mathrm{d}x^i}{\mathrm{d}t}\dfrac{\mathrm{d}x^j}{\mathrm{d}t}=0$ 是定义在流形上的方程.

定义 3.5. *A parametrized curve* $\gamma\colon[a,b]\to M$ *satisfies the equation above is called a geodesic.*

命题 3.6. *Geodesics are parametrited proportionally by arclength*

证明.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(g_{ij}(x(t))\frac{\mathrm{d}x^i(t)}{\mathrm{d}t}\frac{\mathrm{d}x^j(t)}{\mathrm{d}t}\right)&=g_{ij,l}\frac{\mathrm{d}x^l}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}+2g_{ij}\frac{\mathrm{d}^2x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\\&=g_{ij,l}\frac{\mathrm{d}x^l}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}+2g_{ij}\left(-\Gamma_{kl}^i\frac{\mathrm{d}x^k}{\mathrm{d}t}\frac{\mathrm{d}x^l}{\mathrm{d}t}\right)\frac{\mathrm{d}x^j}{\mathrm{d}t}\\&=\left(g_{ij,l}-2g_{kj}\Gamma_{il}^k\right)\frac{\mathrm{d}x^l}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\end{aligned}$$

Claim $g_{ij,l}=g_{kj}\Gamma_{il}^k+g_{ik}\Gamma_{jl}^k$

$$RHS=\frac{1}{2}g_{kj}g^{kp}(g_{pl,i}+g_{ip,l}-g_{il,,p})+\frac{1}{2}g_{ik}g^{kp}(g_{pl,j}+g_{jp,l}-g_{jl,p})$$

$$= \frac{1}{2}(g_{il,i} + g_{ij,l} - g_{il,j}) + \frac{1}{2}(g_{il,j} + g_{ji,l} - g_{jl,i}) = g_{ij,l}$$

□

定理 3.7. $\forall p \in M, \exists \mathcal{U}_{V,\delta} = \{(q,v) \mid p,q \in V \subset M\, open v \in T_qM, \|v\| < \delta, \delta > 0\}$

and a $\varepsilon > 0$ *and* C^∞ *map* $\gamma : (-\varepsilon,\varepsilon) \times \mathcal{U}_{V,\delta} \to M$ s.t. $\forall (q,v) \in \mathcal{U}_{V,\delta}$, *the curve* $t \mapsto \gamma(t,q,v)$ *is the unique geodesic satisfying* $r(0,q,v) = q, r'(0,q,v) = v \in T_qM$

3 月 4 日 22 分 27 秒

引理 3.8 (Homogeneity of geodesic). *If the geodesic* $\gamma(t,q,v)$ *is defined on* $t \in (-\varepsilon,\varepsilon)$, *then the geodesic* $\gamma(t,q,\lambda v)$,$\lambda \in \mathbb{R}^+$ *is defined on the interval* $t \in (-\frac{\varepsilon}{\lambda}, \frac{\varepsilon}{\lambda})$ *and*

$$\gamma(t,q,\lambda v) = \gamma(\lambda t,q,v).$$

废稿

Consider the length functional $L\colon C_{p,q} \to \mathbb{R}$.

我要找 L 的最小值点. 一个简单但关键的观察是：如果 γ 是连接 p 和 q 的最短线，那么它也是连接其上 p,q 之间任意两点的最短线. 因此我们可以将问题局部化！

下一个观察是，作为 L 的我要找 L 的最小值点，首先找 L 的极小值点.

假设 $\gamma_0 \in C_{p,q}$ 是 L 的极小值点，那么对于任意一族曲线 $\gamma_\varepsilon\colon (-\delta,\delta) \to C(p,q)$，都应有

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} L(\gamma_\varepsilon) = 0, \quad \left.\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\right|_{\varepsilon=0} L(\gamma_\varepsilon) \geqslant 0.$$

注记. γ_ε 上得附加可微性吧？不然 $L(\gamma_\varepsilon)$ 怎么可导？

Localizable

Suppose γ is the shortest curve connecting p and q, then it is also the shortest curve connecting any two points on γ between p and q. WLOG, we can suppose p,q are in one coordinate chart.

注记. 但这里是不是还需要说明我们不需要考虑那些跑出 p,q 落在的坐标卡的那些曲线，只考虑包含在坐标卡里的那些曲线.

Energy functional

$$L(\gamma_\varepsilon) = \int_a^b \sqrt{g_{ij}(x\circ\gamma_\varepsilon(t))\frac{\mathrm{d}x^i\circ\gamma_\varepsilon(t)}{\mathrm{d}t}\frac{\mathrm{d}x^j\circ\gamma_\varepsilon(t)}{\mathrm{d}t}}\mathrm{d}t$$

要对它求导太麻烦，为此我们考虑能量泛函 $E(\gamma) = \frac{1}{2}\int_a^b g(\gamma'(t),\gamma'(t))\mathrm{d}t$.

引理 3.9. $\forall\ \gamma \in C_{p,q}$, $\gamma : [a,b] \to M$, *we have*

$$L(\gamma)^2 \leqslant 2(b-a)E(\gamma).$$

and "$=$" *holds iff* $\|\gamma'(t)\| \equiv$ const.

证明.

$$L(\gamma)=\int_a^b\|\gamma'(t)\|\mathrm{d}t\leqslant\left(\int_a^b 1^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_a^b\|\gamma'(t)\|^2\mathrm{d}t\right)^{\frac{1}{2}}=\sqrt{b-a}\sqrt{2E(\gamma)}.$$

□

容易验证 $E(\gamma)$ 只能对于参数化曲线 $\gamma\colon[a,b]\to M$ 定义，这与长度泛函是不同的.

If γ is arclength parametrized, then $L(\gamma)^2=2L(\gamma)\cdot E(\gamma)\Longrightarrow L(\gamma)=2E(\gamma)$.

Let us fix some notations. Suppose

$$\begin{aligned}\gamma:[a,b]\longrightarrow U\subset M^n\overset{x}{\longrightarrow}x(U)\subset\mathbb{R}^n\\ t\longmapsto\gamma(t)\in U\longmapsto x(\gamma(t))=:x(t)\end{aligned}$$

where $\gamma:[a,b]\to M$ is a parametrized curve and (U,x) is a chart.

Given $y\colon[a,b]\to\mathbb{R}^n$ a parametrized curve such that $y(a)=y(b)=0$, define $\gamma_\varepsilon(t)=x(t)+\varepsilon y(t)$. You can belive that for sufficient small δ, γ_ε is contained in $x(U)$, $\forall\,\varepsilon\in(-\delta,\delta)$.

注记. *一个问题是这样构造出来的 γ_ε 是否把所有的这种扰动找全了.*

注记. *流形上没有线性结构，搬到 $\mathbb{R}^n$ 上去加!*

命题 3.10 (光滑 + 最短线 + 平行弧长参数 $\Longrightarrow$ 能量泛函临界点)**.** *If γ is a C^∞ shortest curve from p to q.（前一句话与参数化无关，但后一句话给定了一个参数化）Then γ with a parametrization $\gamma:[a,b]\to U\subset M$ s.t. $\|\gamma'(t)\|\equiv\mathrm{const}$ is a critical point of E,i.e.,$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}E(\gamma_\varepsilon)=0$.*

注记.

- *原则上来说最短线是在所有分段光滑的曲线中找的，以后会说明最短线一定是光滑的.*
- *在不担心这个额外的光滑性假定的条件下，上面的命题告诉我们，最短线赋予平行于弧长的参数一定是能量泛函的临界点.*

  因此如果我们去找能量泛函的临界点，是不会漏掉最短线的.

证明. γ shortest $\Longrightarrow L(\gamma)\leqslant L(\gamma_\varepsilon)$

$L(\gamma)=\sqrt{2(b-a)E(\gamma)}$

$L(\gamma_\varepsilon)\leqslant\sqrt{2(b-a)E(\gamma_\varepsilon)}$

$\Longrightarrow E(\gamma)\leqslant E(\gamma_\varepsilon)$

$\Longrightarrow\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}E(\gamma_\varepsilon)=0.$ □

最短线加弧长参数是临界点，临界点如果都不是弧长参数就完了，没听懂.

4 指数映射

要根据一点附近的测地线的性质，来确定一个坐标系，使得测地线在这个坐标映射下投到欧氏区域后是直线.

其实拿切空间来做坐标区域应该是个挺自然的想法，毕竟切空间是该处的一阶线性近似

$$\begin{aligned}\exp_p : T_pM &\longrightarrow M \\ v &\longmapsto \gamma(1, p, v)\end{aligned}$$

- 选取 1 能够使测地线走的长度等于 $\|v\|_g$.

指数映射的定义域

3 月 4 日 52 分 30 秒

$V_p := \{v \in T_pM \mid \text{the geodesic } \gamma(t, p, v) \text{ is defined on } [0, 1]\}$.

为了 $\exp_p$ 成为坐标映射，我们希望 V_p 至少包含以 O 为心的一个开球！

3 月 4 日 55 分 45 秒

命题 4.1.

(1) V_p *is star-shaped aroud* $O \in T_pM$*, i.e.*$\forall\ v \in V_p, \forall\ \lambda \in [0, 1]$*, then* $\lambda v \in V_p$.

(2) $\forall\ p, \exists\ \varepsilon = \varepsilon(p)$, s.t. $\gamma(t, p, v)$ *is defined on* $[0, 1]$ *once* $\|v\| < \varepsilon$.

3 月 4 日 1 小时 1 分 0 秒，反函数定理

3 月 4 日 1 小时 5 分 2 秒

命题 4.2. $\mathrm{d}\exp_p = \mathrm{Id}_{T_pM}$.

由逆映射定理，存在 p 点的一个邻域 U 使得 $\exp_p^{-1} : U \to T_pM$ 是微分同胚.

距离 $\exp_p^{-1}$ 成为坐标映射只差 T_pM 到 $\mathbb{R}^n$ 的一个同构，任取 T_pM 的一组基即可.

命题 4.3. $\Gamma_{ij}^k(p) = 0$.

命题 4.4. 选取 T_pM 的一组基 $\{v_1, \cdots, v_n\}$. 断言 g 在坐标映射 $\exp_p^{-1} : U \to T_pM \cong \mathbb{R}^n$ 下的分量在 O 处的取值 $g_{ij}(O) = g(v_i, v_j)$.

定义 4.5. 选取 T_pM 的一组标准正交基，此时的 $(\exp_p^{-1}, U)$ 称为 p 的一个法坐标.

3 月 4 日 1 小时 17 分 45 秒

证明. $0 = \dfrac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma_{jk}^i(x(t)) \dfrac{\mathrm{d}x^j}{\mathrm{d}t} \dfrac{\mathrm{d}x^k}{\mathrm{d}t}$ □

极坐标

A curve $c(t) = (r(t), \varphi^1(t), \cdots, \varphi^{n-1}(t))$
$c'(t) = (\frac{\mathrm{d}r}{\mathrm{d}t}, \frac{\mathrm{d}\varphi^1}{\mathrm{d}t}, \cdots, \frac{\mathrm{d}\varphi^{n-1}}{\mathrm{d}t}) =: (v^1, v^2, \cdots, v^n)$
$\|c'(t)\| = g_{ij}(c(t))v^iv^j = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \underbrace{\sum_{i,j=1}^{n-1} g_{\varphi^i\varphi^j}\frac{\mathrm{d}\varphi^i}{\mathrm{d}t}\frac{\mathrm{d}\varphi^j}{\mathrm{d}t}}_{\geqslant 0}$

3 月 8 日第二段 11 分 20 秒

推论 4.6. *For any $p \in M$,$\exists\ \rho > 0$ s.t. $\forall\ q$ with $d(p,q) = \rho$, there exists a unique shortest curve $\in C_{p,q}$.*

证明. $\exists\ \rho > 0$ s.t. $B(p, 2\rho)$ lies in a Riemannian polar coordinate neighborhood.

For any curve $c \in C_{p,q}$

$$c : [0, T] \to M, c(0) = p, c(T) = q$$

□

推论 4.7. 最短线是光滑的.

5 一致邻域

3 月 8 日 25 分 20 秒

3 月 8 日 28 分 56 秒

定义 5.1. *totally normal neighborhood.*

$\forall\ p \in M$, if $W \ni p$, W is a normal neighborhood of every point $q \in W$, then W is cassled a **totally normal neighborhood**.

5.1 totally normal neighborhood 的存在性

3 月 8 日第二段 35 分 4 秒

引理 5.2.

$$\mathrm{d}\exp(p, 0_p) : T_{p,0_p}(TM) \longrightarrow T_{p,p}(M \times M)$$

is non sigular.

3 月 8 日第二段 1 小时 3 分 54 秒

定理 5.3. *For any* $p \in M$, $\exists$ *a neighborhood* W *of* p*, and a* $\delta > 0$ *such that* $\forall q \in W$, $\exp_q$ *is a diffeomorphism on* $B(0_q, \delta) \subset T_q M$ *and*

3 月 8 日第二段 1 小时 14 分 7 秒

推论 5.4.

6 Cut locus 1

3 月 8 日第二段 1 小时 25 分 32 秒，总结

测地线的最大存在区间的端点是开的

3 月 8 日第二段 1 小时 28 分 5 秒

给定 $p \in M, v \in T_pM$，有测地线 $\gamma(t,p,v) = \exp_p tv$.

假设 $[0,b]$ 是 γ 的最大存在区间. 记 $q = \gamma(b,p,v), w = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=b}\gamma(t,p,v)$.

存在经过 q，以 w 为初始切向量的测地线 $\tilde{\gamma}$，$\tilde{\gamma}$ 在某区间 $(-\varepsilon,\varepsilon)$ 上有定义.

断言 $\gamma\big|_{(b-\varepsilon,b]}$ 的反转与 $\tilde{\gamma}\big|_{(-\varepsilon,0]}$ 的反转都是以 q 为起点，以 $-w$ 为初始切向量的测地线.

这是由链式法则与测地线方程的特点保证的.

由存在唯一性知 γ 与 $\tilde{\gamma}$ 在公共定义域上重合. 这与 $[0,b]$ 是 γ 的最大存在区间矛盾.

测地线是最短线的最大区间相对测地线的最大存在区间是闭的

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由最短线也是连接其上任意两点的最短线，知测地线是最短线的点是个区间.

$A = \{t > 0 \mid d(p,\gamma(t)) = t\|v\|_g\}$ 是闭的.

Either $A = (0,b)$ or $A = (0,a]$ for some $0 < a < b$.

定义 6.1.

- 如果 $A = (0,a]$，则称 $\gamma(a)$ 是 p 沿测地线 γ 的割点.
- 如果 $A = (0,b)$，则称 p 沿测地线 γ 没有割点.
- 称割点的全体为 p 的割迹，记作 $C(p)$.
- 定义 τ: $\{v \in T_pM \mid \|v\|_g = 1\} \to \mathbb{R}, \tau(v) = \begin{cases} a & \textit{if } \exp_p(av) \textit{ is a cut point of } p \\ b & \textit{if } p \textit{ has no cut point along } t \mapsto \exp_p(tv) \end{cases}$

定义 6.2.

- *Define a map* $\tau : S_p \to \mathbb{R} \cup \{\infty\}$

  $\forall v \in S_p, \tau(v) = \begin{cases} a & \textit{if } \exp_p(av) \textit{is a cut point ofp} \\ \infty & \textit{ifphas no cut point alongt} \mapsto \exp_p(tv) \end{cases}$

  $E(p) = \{tv \mid v \in S_p, 0 \leqslant t < \tau(v)\}$

  $\tilde{C}(p) = \{tv \mid v \in S_p, t = \tau(v)\}$

  $C(p) = \{\textit{cut points ofp}\} = \exp_p(\tilde{C}(p))$

  $[0,b)$ is the maximal interval on which $t \mapsto \exp_p tv$ is defined.

命题 6.3. $\forall\ p,q \in M, \exists$ *two shortest curve connecting* p *and* q*,*

推论 6.4. $\exp_p : E(p) \to \exp_p(E(P)) \subset M$ *is injective.*

证明. Suppose $\exists V, W \in E(p)$ s.t. $\exp_p(V) = \exp_p(W) = q$.

$t \mapsto \exp_p\left(t\frac{v}{\|v\|}\right)$

$t \mapsto \exp_p\left(t\frac{w}{\|w\|}\right)$

Contradiction. □

推论 6.5. $\exp_p(E(P)) \cap C(p) = \varnothing$.

证明. Suppose $\exists\ v \in \tilde{C}(p), W \in E(p)$ s.t. $\exp_p V = \exp_p W = q$

Contradiction. □

Question: $\exp_p(E(p)) \cup C(p) = M$?

$\mathbb{R}^2 \setminus \{0\}$

$\forall q \in \exp_p(E_p) \cup C(p) = M$?

7 Hopf-Rinow Theorem

3 月 11 日 27 分 14 秒

任给 $p_0, q \in M, d(p_0, q) = r_0$. 我们想要找 p_0, q 之间的最短线.

我们知道局部上总是可以做的，问题是 p_0, q 可能离得很远.

思路是一步一步走.

选取以 p_0 为中心的一个 normal ball $B(p_0, \rho_0)$，若 $q \in B(p_0, \rho_0)$，结束.

若 $q \notin B(p_0, \rho_0)$，假设 p_0, q 之间存在最短线 γ，易知

- $\gamma \cap \partial B(p_0, \rho_0) = \{pt\} =: \{p_1\}$.
- $d(p_0, q) = \min\limits_{p \in \partial B(p_0, \rho_0)} d(p, q)$.

从 p_1 出发，我们可以找一个 normal ball $B(p_1, \rho_1)$，并重复上述操作.

问题是：(1) p_0 到 p_2 的分段曲线是最短的吗？(2) 最终能达到 q 吗？

- $d(p_1, q) = r_0 - \rho_0$
  - 假如 $d(p_1, q) < r_0 - \rho_0$，那么可以找到一条连接 p_0, q 的长度小于 r_0 的曲线，矛盾.
  - 假如 $d(p_1, q) > r_0 - \rho_0$. 任选连接 p_0, q 的曲线 γ，$Length(\gamma) \geqslant \rho_0 + d(p_1, q)$.
    取下确界，得 $r_0 \geqslant \rho_0 + d(p_1, q) > r_0$，矛盾.
- $d(p_0, p_2) = \rho_0 + \rho_1$
  - $d(p_0, p_2) \leqslant d(p_0, p_1) + d(p_0, p_2) = \rho_0 + \rho_1$.
  - $d(p_0, p_2) \geqslant d(p_0, q) - d(p_2, q) = r - (r - \rho_0 - \rho_1) = \rho_0 + \rho_1$.

因此，走了 n 步之后，p_0 和 p_n 之间的连线仍是最短的.

3 月 11 日 55 分 24 秒名场面：方向决定道路，道路决定命运.

容易举出一些例子使得 (2) 不成立，为此我们附加一些额外的条件.

3 月 11 日 59 分 19 秒

定义 7.1.

- *injective radius at* $p \in M$: $i(p) = \sup\left\{\rho > 0 \mid \exp_p\big|_{B(O,\rho)} \text{ is a diffeomorphism}\right\}$.
- *injective radius of* M : $i(M) = \inf\limits_{p \in M} i(p)$.

M compact $\Longrightarrow i(M) > 0$.

3 月 11 日 1 小时 4 分 52 秒

Given $p \in M$,

1. Assumption I: $\overline{B_p(r)}$ is compact($\Longleftrightarrow$All closed bounded subsets of M is compact).
2. Assumption II: (M, g) is a complete metric space.
3. Assumption III: $\exp_p(p)$ is defined on the whole space T_pM.

这三个条件都可以保证 (2). 下面用 Assupmtion III 推 (2).

3 月 21 日 1 小时 11 分 43 秒

证明. $p, V \in T_pM$ $c(t) = \exp_p tV$

Aim:$c(r) = \exp_p(rV) = q$

Consider the set $I := \{t \in [0, r] \mid d(c(t), q) = r - t\}$

□

1 小时 24 分 33 秒

事实上，上面几种假定是等价的，这就是 Hopf-Rinow 定理.

3 月 15 日 2 分 31 秒

定理 7.2 (Hopf-Rinow,1931)**.** *Let (M, g) be a Riemannian manifold, TFAE*

(1) (M, d_g) *is a complete metric space.*

(2) *All closed bounded subsets of M is compact.*

(3) $\exists\ p \in M$*,* $\exp_p$ *is defined on the whole* T_pM*.*

(4) $\forall\ p \in M$*,* $\exp_p$ *is defined on the whole* T_pM*.*

Moreover, each of the statements $(1) - (4)$ *implies*

(5) $\forall\ p, q \in M$ *can be joined by a shortest curve.*

注记. 原始论文：*Ueber den Begriff der vollständigen differentialgeometrischen Fläche*.

证明.

- $(3) \Longrightarrow (2)$

  Claim:$\forall\ r > 0, \overline{B(p, r)}$ is compact.

  For any bounded closed subset K, $\exists\ r_k$ such that $K \subset \overline{B(p, r_k)}$.

  FACT:$\overline{B(p, r)} = \exp_p(\overline{B(Op, r)})$

  - $\exp_p(\overline{B(O_p, r)}) \subset \overline{B(p, r)}$

    $\forall\ v \in \overline{B(O_p, r)}, d(p, \exp_p V) \leqslant r \Longrightarrow \exp_p V \in \overline{B(p, r)}$
  - $\forall q \in \overline{B(p, r)}$,

- $(2) \Longrightarrow (1)$

- $(1) \Longrightarrow (4)$

  Suppose $\exists\ p \in M$ and $v \in T_pM$ such that the geodesic $t \mapsto \exp_p tv$ is defined on the maximal interval $[a, b), b < \infty$.

  For any $\{t_n\} \subset [a, b)$ such that $t_n \to b$, $d(\exp_p t_n v, \exp_p t_m v) \leqslant \|v\|_g |t_n - t_m|$ and then $\{\exp_p t_n v\}$ is a Cauchy sequence.

  $\exists\ p_0 \in M, \lim\limits_{n \to +\infty} \exp_p t_n v = p_0$,i.e. $\forall \delta > 0, \exists N$ such that $\exp_p t_n v \in B(p_0, \delta), \forall n \geqslant N$.

□

引理 7.3. 内容...

8 Cut locus 2

3 月 15 日 39 分 24 秒

定理 8.1. *Let (M, g) be a complete Riemannian manifold, then*

$$M = \exp_p(E(p)) \sqcup c(p).$$

定理 8.2. *Let (M, g) be a complete Riemannian manifold.*

Let $\gamma \colon [a, b] \to M$ be a normal geodesic with $p = \gamma(0)$, v

证明. Choose a sequence of parameters

$$a_1 > a_2 > a_3 > \cdots, \quad \lim_{i \to +\infty} a_i = a.$$

By completeness, $\exists v_i \in T_pM, \|v_i\| = 1$ such that

$$\gamma_i(t) = \exp_p tv_i, t \in [0, b_i]$$

is a shortest curve form p to $\gamma(a_i)$, where $b_i = d(p, \gamma(a_i))$.

Notice that $v_i \neq v$.

$\lim_{i \to +\infty} p_i =$ □

9 黎曼覆盖映射

10 Existence of shortest curves in given homotopy class

- isometry：微分同胚，度量等于拉回
-

3 月 15 日 1 小时 26 分 15 秒

定理 10.1. *Let (M, g) be compact.*

Then every homotopy class of closed curves in M contains a curve which is shortest in its homotopy class and a geodesic.

引理 10.2. *Let (M, g) be compact, $\exists \rho_0 > 0$ such that for any $\gamma_0, \gamma_1 \colon S^1 \to M$ be closed curves with $d(\gamma_0(t), \gamma_1(t)) \leqslant \rho_0, \forall\ t \in S_1$ we have γ_0 and γ_1 are homotopic.*

引理 10.3. *A shortest curve in a homotopy class is geodesic.*

证明. Let $(\gamma)n_{n\in\mathbb{N}}$ is a minimizing sequence for length in the homotopy class.

All are parametrized proportional to arc length.

We can find $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 2\pi$ with the property that

$$Length(\gamma_n\ |_{t_i, t_{i+1}}) \leqslant \frac{\rho_0}{2}$$

□

11 title

11.1 前情回顾

Riemannian Covering map

$\pi : (\tilde{M}, \pi^* g) \to (M, g)$ smooth map

locally Riemannian isometry

isometry $\varphi\colon (M, g_M) \to (N, g_N)$ is called an isometry if φ is diffeomorphism and $g_M = \varphi^* g_N$

locally isometry $\varphi\colon (M, g_M) \to (N, g_N)$ smooth map, $\forall p \in M \exists U \in p$such that $\varphi\big|_U : U \to \varphi(U)$ is an isometry 问题：对 $\varphi(U)$ 有没有要求

locally Riemannian isometry $\varphi\colon (M, g_M) \to (N, g_N)$ smooth map, $\forall\ p \in M, \mathrm{d}\varphi_p : T_pM \to T_{\varphi(p)}N$ is a linear isometry.

命题 11.1. *Let $\varphi\colon (M, g_M) \to (N, g_N)$ be a locally Riemannian isometry.*

(1) *φ maps geodesics to geodesics.*

(2) *For any $\tilde{p}, \tilde{v} \in T_{\tilde{p}}M$, we have*

$$\varphi \circ (\exp_{\tilde{p}} \tilde{v}) = \exp_{\varphi(\tilde{p})}(\mathrm{d}\varphi_{(\tilde{p})}(\tilde{v})).$$

$$\begin{array}{ccc} T_{\tilde{p}}M & \xrightarrow{\mathrm{d}\varphi(\tilde{p})} & T_{\varphi(\tilde{p})N} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad\varphi\quad} & N \end{array}$$

(3) *φ is distance non-increasing.*

$$\forall\ \tilde{p},\tilde{q}, d_N(\varphi(\tilde{p}),\varphi(\tilde{q})) \leqslant d_M(\tilde{p},\tilde{q})$$

(4) *φ is bijective, then it is distance preserving.*

定理 11.2. *(M, g_M) complete Riemannian manifold, $p, q \in M$. Every homotopy class of paths from p to q contains a shortest curve.*

证明. Assume that (M, g_M) complete$\Longrightarrow (\tilde{M}, \pi^* g)$ is complete. □

命题 11.3. *Let $\pi\colon (\tilde{M},\tilde{g}) \to (M,g)$ is a Riemannian covering map, then (M,g) complete iff $(\tilde{M},\tilde{g})$ complete.*

证明.

ete $\Longrightarrow (\tilde{M},\tilde{g})$complete $\forall \tilde{p} \in \tilde{M}, \tilde{v} \in T_{\tilde{p}\tilde{M}}, t \mapsto \exp_{\tilde{p}} t\tilde{v}$

$p = \pi(\tilde{p}), v = \mathrm{d}\pi(\tilde{p})(\tilde{v})$

geodesic $t \mapsto \exp_p tv$ is defined on $[0,\infty)$

path liftying, $\exists \tilde{\gamma}$ a path in $\tilde{M}$ such that $\tilde{\gamma(0)} = \tilde{p}, \pi \circ \tilde{\gamma} = \gamma$

$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t}\Big|_{t=0} = \tilde{v}$

ete $\Longrightarrow (M,g)$complete $\forall p \in M, \forall v \in T_pM, t \mapsto \exp_p tv$

□

命题 11.4. *Let $\pi\colon (\tilde{M},\tilde{g}) \to (M,g)$ is a local Riemannian isometry. Suppose $(\tilde{M},\tilde{g})$ complete. Then (M,g) is complete and π is a Riemannian covering map.*

证明. (1) π is surjective.

$\forall \tilde{p} \in \tilde{M}, p = \pi(\tilde{p}) \in M$

$\forall q \in M$,$\exists$ a shortest geodesic γ from p to q.

Let $\tilde{\gamma}$ be the lifting of γ starting at $\tilde{p} = \tilde{\gamma}(0)$

$\pi \circ \tilde{\gamma} = \gamma, q = \gamma(t_0), \pi \circ \tilde{\gamma}(t_0) = \gamma(t_0) = q$

(2) evenly covered

$p \in U, \pi^{-1}(U) = \bigsqcup_{\alpha\in\Lambda} \tilde{U}_\alpha$

$\pi\colon \tilde{U}_\alpha \to U$ diffeomorphism

Normall ball $B(p,\varepsilon)$

$\tilde{U}_\alpha = B(\tilde{p}_\alpha,\varepsilon)$ metric ball

(a) $\tilde{U}_\alpha \cap \tilde{U}_\beta = \varnothing, \forall \alpha \neq \beta$

$d(\tilde{p}_\alpha, \tilde{p}_\beta) \geqslant 2\varepsilon$

(b) $\pi^{-1}(U) = \bigcup_{\alpha \in \Lambda} \tilde{U}_\alpha$

- $\forall \tilde{q} \in \tilde{U}_\alpha$ for some $\alpha \in \Lambda$
  $\exists$ a geodesic $\tilde{\gamma}$ of length $< \varepsilon$ from

□

(U, x)

$$\frac{\mathrm{d}^2 x^i(t)}{\mathrm{d}t^2} + \Gamma^i_{jk}(x(t))\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} = 0, i = 1, \cdots, n$$

12 能量泛函的变分 II

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}s} &= \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle \mathrm{d}t && \frac{\partial F}{\partial t}\text{视作沿曲线}F(t,\cdot)\text{的向量场}\\ &= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial s}}\frac{\partial F}{\partial t}, \frac{\partial F}{\partial t}\right\rangle \mathrm{d}t && \frac{\partial F}{\partial t}\text{视作沿}F\text{的向量场}\\ &= \int_a^b \left\langle \nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right\rangle \mathrm{d}t\\ &= \int_a^b \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\right\rangle \mathrm{d}t - \int_a^b \left\langle \frac{\partial F}{\partial s}, \nabla_{\frac{\partial}{\partial t}}\frac{\partial F}{\partial t}\right\rangle \mathrm{d}t\\ E'(0) &= \langle V, T\rangle\Big|_a^b - \int_a^b \langle V, \nabla_T T\rangle\,\mathrm{d}t\end{aligned}$$

3 月 29 日 1 小时 35 分 41 秒

12.1 Gauss 引理

3 月 29 日 1 小时 36 分 33 秒

12.2 第二变分公式

3 月 29 日 1 小时 51 分 30 秒

Chapter 3

联络和曲率

1 仿射联络

为什么要对向量场求导？因为欧氏空间中我们就已经在对向量场求导了.

比如浸入 $\gamma\colon I \to \mathbb{R}^3$ 的曲率, 我们选择弧长参数，$\gamma'(s)$ 是一个沿 γ 的单位长的切向量场，我们继续对它求导，这就是对向量场的求导，得到 $\gamma''(s)$，我们把这玩意的模长叫做曲率，用来刻画曲线的弯曲程度.

2 拉回丛及诱导联络

定义 2.1. 拉回丛

例子 2.2. 沿曲线的向量场

定义 2.3. 设 $\varphi\colon N\to M$ 是光滑映射. 称 $V\colon N\to TM$ 是沿 φ 的向量场, 如果 $V(x)\in T_{\varphi(x)}M$.

命题 2.4. 两个概念是一致的.

定义 2.5. 诱导联络

命题 2.6. 诱导联络在基下的计算方式.

2.1 沿曲线的协变导数

定义 2.7. 给定 C^∞ 曲线 $c\colon[a,b]\to M$，称 $V\colon[a,b]\to M$ 是沿 c 的向量场，如果 $V(t)\in T_{c(t)}M$. 我们称 V 是光滑的，如果对任意的 $f\in C^\infty(M)$，函数 $V(t)f$ 是光滑的. 记沿 c 的光滑向量场全体为 $\Gamma(TM|_c)$，它显然是 $C^\infty([a,b])$ 模.

在一个坐标卡 (U,x) 中，$V(t)$ 能够被表达为

$$V(t)=V^i(t)\frac{\partial}{\partial x^i}\bigg|_{c(t)}.$$

容易验证 V 是光滑的 $\Longleftrightarrow$ $V^i(t)$ 是光滑的，$1\leqslant i\leqslant n$.

命题 2.8. 设 M 是光滑流形，∇ 是其上的仿射联络. 存在唯一的 $\dfrac{D}{\mathrm{d}t}\colon\Gamma(TM|_c)\to\Gamma(TM|_c)$ 满足

(1) $\dfrac{D(V+W)}{\mathrm{d}t}=\dfrac{DV}{\mathrm{d}t}+\dfrac{DW}{\mathrm{d}t}$

(2) $\dfrac{D(fV)}{\mathrm{d}t}=\dfrac{\mathrm{d}f}{\mathrm{d}t}V+f\dfrac{DV}{\mathrm{d}t},\forall\ f\in C^\infty([a,b])$

(3) 如果存在 $X\in\mathfrak{X}(M)$ 使得 $V(t)=X(c(t))$，那么 $\dfrac{DV}{\mathrm{d}t}=\nabla_{c'(t)}X$.

2.2 诱导联络

(M,∇)

$\dfrac{DV}{\mathrm{d}t}$,V vector field along a curve c

induced connection

$c\colon(-\varepsilon,\varepsilon)\to M$

Let $\varphi\colon N\to M$ C^∞ map.

A C^∞ vector field along φ.

$x\in N\mapsto V(x)\in T_{\varphi(x)}M$

$\varphi(x)\in M$, frame field E_i in a neighborhood

$V(x)=\sum V^i(x)E_i(\varphi(x))$ 其中 V^i 看作 N 上的函数

Given $u\in T_xN$, $\tilde{\nabla}_uV=\sum u(V^i)E_i(\varphi(x))+V^i(x)\nabla_{\mathrm{d}\varphi(x)(u)}E_i(\varphi(x))$

induced connection

更多内容可参考刘老师 18 年的某次作业

3 平行移动

上节的铺垫使得我们能够讨论平行性.

定义 3.1. 称沿曲线 $c\colon [a,b] \to M$ 的向量场 V 是平行的如果 $\dfrac{DV}{\mathrm{d}t} \equiv 0$.

3 月 22 日第二段 8 分 8 秒

给定一个切向量 $V_a \in T_{c(a)}M$，假使能够找到一个沿 c 的平行切向量场 V 满足 $V(a) = V_a$，我们便可认为 V_a 沿曲线 c 平行地移动到了 $c(t)$ 处成为 $V(t)$.

我们问 V 是否存在？在多大的范围内存在？这等价于去解方程 $\dfrac{DV}{\mathrm{d}t} = 0$.

回忆测地线方程是一个非线性 ODE，因此我们只有解的局部存在性. 而这里 $\dfrac{DV}{\mathrm{d}t} = 0$ 是一个线性方程，从而可保证解整体存在.

命题 3.2. 设 $c\colon [a,b] \to M$ 是光滑曲线. 设 $V_0 \in T_{c(t_0)}M$, $t_0 \in [a,b]$，那么存在唯一的沿 c 平行的向量场 V 使得 $V(t_0) = V_0$.

证明. $c(I) \subset (U,x)$ □

3 月 22 日第二段 14 分 35 秒

命题 3.3. *Let c be a C^∞ curve with $c(0) = p, c'(0) = X(p)$*

Let $Y \in \Gamma(TM)$

Then $\nabla_{X(p)}Y = \lim\limits_{h\to 0} \dfrac{P^{(}_{c,0,h}Y(c(h))) - Y(c(0))}{h}$

证明. Let $V_1, \cdots, V_n$ be parallel vector fields along c which is linearly independent.

$Y(c(t)) = f^i(t)V_i(t)$，这是一件非常方便的事情

$RHS = \lim\limits_{h\to 0} \dfrac{f_i(h)V_i(0) - f^i(0)V_i(0)}{h} = \dfrac{\mathrm{d}f^i}{\mathrm{d}h}\Big|_{h=0} V_i(0)$

$= \dfrac{D}{\mathrm{d}t}(f^i(t)V_i(t))\Big|_{t=0}$

$= \dfrac{DY}{\mathrm{d}t}(0) = \nabla_{\frac{\mathrm{d}c}{\mathrm{d}t}(0)}Y = \nabla_{X(p)}Y$ □

4 张量场的协变导数

本节我们从切丛上的一个联络出发, 定义张量丛上的一个联络.

定理 4.1. 设 M 是光滑流形，∇ 是其上的仿射联络，那么存在唯一的映射

$$\nabla\colon \Gamma(TM)\times\Gamma\left(\bigotimes^{r,s}TM\right)\to\Gamma\left(\bigotimes^{r,s}TM\right)$$

满足

(1) $\nabla_{fX+fY}A=f\nabla_X A+g\nabla_Y A$

(2) $\nabla_X(A_1+A_2)=\nabla_X A_1+\nabla_X A_2$

(3) $\nabla_X(fA)=(Xf)A+f\nabla_X A$

(4) 当 $A\in C^\infty(M)$ 或 $\Gamma(TM)$ 时，∇ 与给定的仿射联络一致.

(5) $\nabla_X(A_1\otimes A_2)=(\nabla_X A_1)\otimes A_2+A_1\otimes\nabla_X A_2$

(6) $C(\nabla_X A)=\nabla_X(CA)$，其中 $C\colon\Gamma(\bigotimes^{r,s}TM)\to\Gamma(\bigotimes^{r-1,s-1}TM)$

证明. $A\in\Gamma(\bigotimes^{r,s}TM)$
$A=A^{i_1i_2\cdots i_r}_{j_1j_2\cdots j_s}Y_{i_1}\otimes Y_{i_2}\otimes\cdots\otimes Y_{i_r}\otimes\omega^{j_1}\otimes\cdots\otimes\omega^{j_s}$
$\nabla_X A=\sum\nabla_X$
$=\sum X(A^{i_1\cdots i_r}_{j_1\cdots j_s})Y_{i_1}$
线性，Leibniz
唯一的问题是如何对微分 1 形式求导
$\omega\in\Omega^1(M)=\Gamma(T^*M)$
$\nabla_X\omega$?
$\forall Y\in\Gamma(TM),\omega(Y)\in C^\infty(TM)$
$X(\omega(Y))=\nabla_X(\omega(Y))=\nabla_X(C(\omega\otimes Y))=C(\nabla_X(\omega\otimes Y))$
$C(\nabla_X\omega\otimes Y+\omega\otimes\nabla_X Y)$
$\nabla_X(\omega)Y+\omega(\nabla_X Y)$
$(\nabla_X\omega)=X(\omega(Y))-\omega(\nabla_X Y)$
$\Longrightarrow$ uniqueness □

注记. (1) *is a consequence of the other assumptions.*
不是那么令人惊讶，这是说在这里是多余的，而不是在仿射联络的最初定义中也是多余的
$\forall X,Y,Z\in\Gamma(TM)$
$f,g\in C^\infty(TM),\omega\in\Gamma(T^*M)$

证明. $(fX+gY)\omega(Z)=\nabla_{fX+gY}\omega(Z)+\omega(\nabla_{fX+gY})Z$
$=fX(\omega(Z))+gY(\omega(Z))$ □

推论 4.2. $\forall\ A \in \Gamma(\bigotimes^{r,s} TM), \omega_\alpha \in \Gamma(T^*M), \alpha = 1, 2, \cdots, r, Y_j \in \Gamma(TM), j = 1, \cdots, s$

We have $(\nabla_X A)(\omega_1, \cdots, \omega_s; Y_1, \cdots, Y_s)$

$= A(\omega_1, \cdots, \omega_r, Y_1, \cdots, Y_s)$

locality

$\nabla_X A(p)$ only depends on X at p and Y in $U \ni p$.

(M, ∇)

$\varphi\colon V \to W$ isomorphism

$\varphi^*\colon W^* \to V^*$ isomorphism ,$\alpha \mapsto \varphi^*(\alpha)$

$\forall v \in V, \varphi^*(\alpha)(v) := \alpha(\varphi(v))$

$P_{c,0,t}\colon T_{c(0)}M \to T_{c(t)}M$

$\longrightarrow \tilde{P}_{c,0,t}\colon \bigotimes^{r,s} T_{c(0)}M \to \bigotimes^{r,s} T_{c(t)}M$

$v_1 \otimes \cdots \otimes v_r \otimes \omega^1 \otimes \cdots \otimes \omega^r \mapsto P_{c,0,t}(v_1) \otimes \cdots \otimes$

Define $\nabla_{X(p)} A := \lim_{h \to 0} \frac{\tilde{P}}{}$

定义 4.3. *A tensor field is called prarllel if* $\nabla_X A = 0$, $\forall X \in \Gamma(TM)$.

$$
\begin{aligned}
&c(t) = (c^1(t), \cdots, c^n(t)) \\
&\frac{Dc'(t)}{\mathrm{d}t} = \frac{D}{\mathrm{d}t}\left(\frac{\mathrm{d}c^i(t)}{\mathrm{d}t}\frac{\partial}{\partial x^i}\right) \\
&= \frac{\mathrm{d}^2 c^i(t)}{\mathrm{d}t^2}\frac{\partial}{\partial^i} + \frac{\mathrm{d}c^i(t)}{\mathrm{d}t}\nabla_{\frac{\mathrm{d}c^j}{\mathrm{d}t}\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} \\
&= \left(\frac{\mathrm{d}^2 c^k(t)}{}\right)\frac{\partial}{\partial x^k}
\end{aligned}
$$

5 Levi-Civita 联络

定义 5.1. 称仿射联络 ∇ 是 (M,g) 上的 Levi-Civita 联络如果 $\nabla_X Y - \nabla_Y X = [X,Y]$ 且 $\nabla g = 0$.

命题 5.2. ∇ 是无挠的 $\iff \Gamma_{ij}^k = \Gamma_{ji}^k$.

证明. 容易看出 $\Gamma_{ij}^k = \Gamma_{ji}^k \iff \nabla_{\partial_i}\partial_j = \nabla_{\partial_j}\partial_i$. 那么

$$\begin{aligned}\nabla_X Y &= \nabla_{X^i\partial_i}\left(Y^j\partial_j\right)\\ &= X^i\frac{\partial Y_j}{\partial x^i}\frac{\partial}{\partial x^j} + X^iY^j\nabla_{\partial_i}\partial_j\\ &= \nabla_Y X + [X,Y].\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijkl} =& \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})\\ & - g^{pq}R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql}\end{aligned}$$

□

命题 5.3. ∇ 是度量相容的 $\iff$ $g_{ij,l} = g_{ik}\Gamma_{jl}^k + g_{kj}\Gamma_{jl}^k$.

证明. □

定理 5.4 (黎曼几何基本定理)**.** 任意黎曼流形 (M,g) 上存在唯一的 *Levi-Civita* 联络.

局部坐标下的证明. 假定存在性. 轮换一下就能说明 $\Gamma_{ij}^k = \frac{1}{2}g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l})$. □

不用坐标的证明. Suppose existence.

Given $X, Y \in \Gamma(TM)$, we can determine $\nabla_X Y$ by determine $\langle\nabla_X Y, Z\rangle$ for any $Z \in \Gamma(TM)$.

$$\begin{aligned}\langle\nabla_X Y, Z\rangle &\overset{(2)}{=} X\langle Y,Z\rangle - \langle Y,\nabla_X Z\rangle\\ &\overset{(1)}{=} X\langle Y,Z\rangle - \langle Y,\nabla_Z X + [X,Z]\rangle\\ &= X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - \langle Y,\nabla_Z X\rangle\\ &\overset{(2)}{=} X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle\nabla_Z Y, X\rangle\\ &\overset{(1)}{=} X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle\nabla_Y Z + [Z,Y], X\rangle\\ &= X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle X,[Z,Y]\rangle + \langle\nabla_Y Z, X\rangle\\ &\overset{(2)}{=} X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle X,[Z,Y]\rangle + Y\langle Z,X\rangle - \langle Z,\nabla_Y X\rangle\\ &\overset{(1)}{=} X\langle Y,Z\rangle - \langle Y,[X,Z]\rangle - Z\langle Y,X\rangle + \langle X,[Z,Y]\rangle + Y\langle Z,X\rangle - \langle Z,\nabla_X Y + [Y,X]\rangle\\ 2\langle\nabla_X Y, Z\rangle &= X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\end{aligned}$$

□

引理 5.5. 设 (M,g) 是黎曼流形，∇ 是其上与 g 相容的联络. 设 $c\colon (a,b)\to M$ 是光滑曲线，$\frac{\mathrm{D}}{\mathrm{d}t}$ 是 ∇ 诱导的沿曲线的协变导数. 设 $V(t), W(t)$ 是沿 c 的光滑曲线，那么

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V(t), W(t)\rangle = \left\langle\frac{DV}{\mathrm{d}t}(t), W(t)\right\rangle + \left\langle V(t), \frac{DW}{\mathrm{d}t}(t)\right\rangle.$$

证明. 设在坐标邻域 (U,x) 中 $V(t)=V^i(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}, W(t)=W^j(t)\frac{\partial}{\partial x^j}\Big|_{c(t)}$, 那么

$$\begin{aligned}
\mathrm{LHS} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(V^i(t)W^j(t)\left\langle \frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle\right) \\
&= \left\langle \frac{\mathrm{d}V^i}{\mathrm{d}t}(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}, W(t)\right\rangle + \left\langle V(t), \frac{\mathrm{d}W^j}{\mathrm{d}t}(t)\frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle + V^i(t)W^j(t)\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle \\
&= \left\langle \frac{\mathrm{d}V^i}{\mathrm{d}t}(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}, W(t)\right\rangle + \left\langle V(t), \frac{\mathrm{d}W^j}{\mathrm{d}t}(t)\frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle + V^i(t)W^j(t)\frac{\mathrm{d}}{\mathrm{d}t}g\left(\frac{\partial}{\partial x^i}\Big|_{c(t)}, \frac{\partial}{\partial x^j}\Big|_{c(t)}\right) \\
&= \left\langle \frac{\mathrm{d}V^i}{\mathrm{d}t}(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}, W(t)\right\rangle + \left\langle V(t), \frac{\mathrm{d}W^j}{\mathrm{d}t}(t)\frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle + V^i(t)W^j(t)c'(t)g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \left\langle \frac{\mathrm{d}V^i}{\mathrm{d}t}(t)\frac{\partial}{\partial x^i}\Big|_{c(t)}, W(t)\right\rangle + \left\langle V(t), \frac{\mathrm{d}W^j}{\mathrm{d}t}(t)\frac{\partial}{\partial x^j}\Big|_{c(t)}\right\rangle + V^i(t)W^j(t)\nabla_{c'(t)}g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \\
&= \left\langle \frac{DV}{\mathrm{d}t}, W\right\rangle + \left\langle V, \frac{DW}{\mathrm{d}t}\right\rangle
\end{aligned}$$

□

注记.

命题 5.6. 设 (M,g) 是黎曼流形，∇ 是其上的仿射联络. 那么 ∇ 与 g 相容当且仅当任意平行移动是等距同构.

证明. $c\colon [a,b]\to M$ curve

$\mathcal{P}_{c,a,t}\colon T_{c(a)}M\to T_{c(t)}M$

-
- 任意 $X,Y,Z\in\Gamma(TM), \forall p\in M$

□

命题 5.7. 设 ∇ 是 M 上的无挠联络. 设 $s\colon \mathbb{R}^2\to M\in C^\infty$，$V$ 是沿 s 的光滑向量场. 那么

$$\widetilde{\nabla}_{\frac{\partial}{\partial x}}s_*\frac{\partial}{\partial y}=\widetilde{\nabla}_{\frac{\partial}{\partial y}}s_*\frac{\partial}{\partial x}.$$

证明. 直接在局部坐标下计算. □

6 能量泛函的第二变分公式与曲率张量

7 协变微分与 Ricci 恒等式

4 月 1 日 1 小时 23 分 16 秒

7.1 局部坐标下的协变微分

8 音乐同构

$$\flat\colon T_pM \longrightarrow T_p^*M, \quad X \longmapsto \flat X, \quad \flat X(Y) = g(X,Y).$$

$$\sharp\colon T_p^*M \longrightarrow T_pM, \quad \omega \longmapsto \sharp\omega, \quad g(\sharp\omega, Y) = \omega(Y).$$

9 算符

9.1 Hessian

9.2 散度

9.3 梯度

-

9.4 拉普拉斯

10 Bianchi 恒等式

- 第一 Bianchi 恒等式的 global 版本、证明和局部版本
- 第二 Bianchi 恒等式的 global 版本、证明和局部版本

命题 10.1. 设 M 是光滑流形，∇ 是其上无挠的仿射联络，R 是相应的的曲率张量. 那么对于任意 $X, Y, Z, W \in \Gamma(TM)$，我们有

(1)（第一 *Bianchi* 恒等式）$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$

(2)（第二 *Bianchi* 恒等式）$(\nabla_X R)(Y,Z)W$

11 Riemann 曲率张量

12 截面曲率

13 高斯绝妙定理

14 Ricci 曲率

15 数量曲率

16 Bochner 公式

17 测地曲率

18 Gauss-Bonnet 公式

定理 18.1. 设 M 是紧的二维黎曼流形, 其边界 ∂M 光滑, 则

$$\int_M K\mathrm{d}A + \int_{\partial M} k_g \mathrm{d}s = 2\pi\chi(M).$$

Chapter 4

Jacobi 场

1 Jacobi 场

定义 1.1. 设 $\gamma\colon[a,b]\to M$ 是一条测地线. 对于 $t_0,t_1\in[a,b]$，如果存在沿 γ 的不恒为零的 *Jacobi* 场 $U(t)$，满足 $U(t_0)=U(t_1)=0$，则称 t_0,t_1 是沿 γ 的共轭值. 将所有这样的 *Jacobi* 场与恒为零的向量场所构成的线性空间的维数称作 t_0 和 t_1 作为共轭值的重数. 称 $\gamma(t_0)$ 和 $\gamma(t_1)$ 为沿 γ 的共轭点.

2 Morse 指标定理

3 Cartan-Hadamard 定理

4 空间形式

定义 4.1. 称常截面曲率的完备黎曼流形为空间型.

引理 4.2. 内容...

定理 4.3. 设 (M_i^n, g_i) 是单连通、截面曲率为 c 的空间型. 设 $p_i \in M_i$，$\left\{e_i^1, \cdots, e_i^n\right\}$ 是 $T_{p_i}M_i$ 的标准正交基，那么存在唯一的保距映射 $\varphi\colon M_1 \to M_2$ 使得 $\varphi(p_1) = p_2, \varphi_{*,p}(e_1^j) = e_2^j$.

5 单连通空间形式的等距群

5.1 $\mathbb{R}^n$

命题 5.1. $Iso(\mathbb{R}^n) \cong T(n) \rtimes O(n)$.

证明. 假设 $f \in Iso(\mathbb{R}^n)$ 满足 $f(0)=0$，否则考虑 $\tilde{f} = f - f(0)$.

(1) f 保持内积. 因为 f 保持距离，所以对任意 $x, y \in \mathbb{R}^n$，有

$$\|f(x) - f(y)\|^2 = \|x - y\|^2 \Longrightarrow \langle f(x), f(y)\rangle = \langle x, y\rangle .$$

(2) f 是线性的.

- $\|f(ax) - af(x)\|^2 = \|f(ax)\|^2 + \|af(x)\|^2 - 2\langle f(ax), af(x)\rangle = \|ax\|^2 + \|ax\|^2 - 2\|ax\|^2 = 0.$
- $\|f(x+y) - f(x) - f(y)\|^2 \overset{\text{展开}}{=\!=\!=} \cdots \overset{\text{脱}f}{=\!=\!=} \cdots \overset{\text{合并}}{=\!=\!=} \|x + y - x - y\|^2 = 0.$

(3) $f \in O(n)$.

□

注记. 证明了稍稍强一点的事：等距 $\Longrightarrow$ 双射.

5.2 $\mathbb{S}^n$

命题 5.2. $Iso(\mathbb{S}^n) \cong \mathrm{O}(n+1)$.

证明. https://math.stackexchange.com/questions/130193/isometries-of-mathbbsn □

5.3 $\mathbb{H}^n$

6 Killing-Hopf 定理

7 距离函数

Chapter 5

比较定理

1 Sturm 比较定理

2 Rauch 比较定理

3 Hessian 比较定理

4 Laplacian 比较定理

5 体积比较定理

Chapter 6

规范理论